

Hierarchical Joint Remote State Preparation in Noisy Environment

Chitra Shukla^{a,*}, Kishore Thapliyal^{b,†}, Anirban Pathak^{b,‡}

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^aGraduate School of Information Science, Nagoya University, Furo-cho 1, Chikusa-ku, Nagoya, 464-8601, Japan

^bJaypee Institute of Information Technology, A-10, Sector-62, Noida, UP-201307, India

Abstract

A novel scheme for quantum communication having substantial applications in practical life is designed and analyzed. Specifically, we have proposed a hierarchical counterpart of the joint remote state preparation (JRSP) protocol, where two senders can jointly and remotely prepare a quantum state. One sender has the information regarding amplitude, while the other one has the phase information of a quantum state to be jointly prepared at the receiver's port. However, there exists a hierarchy among the receivers, as far as powers to reconstruct the quantum state is concerned. A 5-qubit cluster state has been used here to perform the task. Further, it is established that the proposed scheme for hierarchical JRSP (HJRSP) is of enormous practical importance in critical situations involving defense and other sectors, where it is essential to ensure that an important decision/order that can severely affect a society or an organization is not taken by a single person, and once the order is issued all the receivers don't possess an equal right to implement it. Further, the effect of different noise models (e.g., amplitude damping (AD), phase damping (PD), collective noise and Pauli noise models) on the HJRSP protocol proposed here is investigated. It is found that in AD and PD noise models a higher power agent can reconstruct the quantum state to be remotely prepared with higher fidelity than that done by the lower power agent(s). In contrast, the opposite may happen in the presence of collective noise models. We have also proposed a scheme for probabilistic HJRSP using a non-maximally entangled 5-qubit cluster state.

Keywords: Joint remote state preparation, hierarchical quantum communication, amplitude damping noise, phase damping noise, collective noise, Pauli noise, quantum communication.

1 Introduction

There are two major facets of quantum information science: quantum computing [1, 2, 3] and quantum communication [3, 4, 5, 6]. In the last three decades, both of these have attracted major attraction of the scientific community for their ability to perform computational or communication tasks beyond the capacity of their classical counterparts. Specifically, a quantum algorithm may perform a computational task much faster than its classical counterpart. For example, Grover's algorithm [1] can search unsorted database quadratically faster than its best known classical counterpart, and Shor's algorithm [2] can factorize large numbers in a speed not attainable in a classical computer. Similarly, in the domain of quantum communication, protocols have been proposed for teleportation [5], which does not have any classical analogue, and unconditional security of quantum key distribution [4] has been established [7]. Unconditional security is a desirable feat not achievable in classical world. The nonclassical nature of quantum communication schemes drew considerable attention of the scientific community. Especially, teleportation (a communication scheme, in which the state to be transmitted from the sender to the receiver never exists in the channel joining the receiver and the sender) drew major attention because of its magical characteristics. Initially, a teleportation scheme was proposed by Bennett et al. in 1993 [5]. In the original teleportation scheme, the sender Alice used to transmit/teleport an unknown single-qubit quantum state to the receiver Bob using a shared entangled state and two bits of classical communication. Subsequently, several variants of teleportation

*email: shukla.chitra@i.mbox.nagoya-u.ac.jp

†email: tkishore36@yahoo.com

‡email: anirban.pathak@jiit.ac.in

have been proposed. For example, schemes were proposed for quantum information splitting (QIS) or controlled teleportation (CT) [8, 9], quantum secret sharing (QSS) [10], hierarchical quantum information splitting (HQIS) [11, 12], remote state preparation (RSP) [13], etc. (see Ref. [6] for a review). All these schemes can be viewed as variants of teleportation.

Recently, a few hierarchical versions of already existing aspects of quantum communication (variants of teleportation) have been proposed. Specifically, hierarchical quantum information splitting (HQIS) [11, 12, 14, 15], hierarchical quantum secret sharing (HQSS) [12], hierarchical dynamic quantum secret sharing (HDQSS) [16], etc., have been proposed in the recent past. It is also shown that these schemes have enormous practical importance (for a detailed discussion on the interesting applications of these schemes see Sec. 1 of Refs. [12, 16]). In these protocols, there is a hierarchy among the powers of receivers (agents) to reconstruct a quantum state sent by the sender, i.e., the agents are graded in accordance to their power for the reconstruction of an unknown quantum state. Specifically, in HQIS the receivers can reconstruct the teleported quantum state with the help of other receivers (as in QIS [8, 9]), where the power of a particular receiver is decided by the minimum number of receivers required to cooperate with him to enable him to reconstruct the state [12]. In probabilistic HQIS, the same task is performed probabilistically. HQSS scheme can be viewed as a direct application of HQIS where the sender wishes to send an information in pieces to all the receivers who can reconstruct it with the help of either some or all other agents. Later the scheme was extended to propose a HDQSS scheme, and in HDQSS scheme an additional feature to add and drop an agent was included [16], and that made HDQSS most practical hierarchical scheme proposed until now. This is so because, in a practical situation, an agent may resign from a company or take leave, and the company may decide to recruit a new agent as his/her replacement. Further, if the sale of the company gets increased it may recruit a new agent. However, these recently introduced schemes of hierarchical quantum communication lacks a particularly important feature. In all the existing schemes, there is only one sender, but in many practical purposes (as illustrated with an example in Sec. 3) we need more than one sender for the secrecy of the initial message/state to be transmitted in a hierarchical manner. This paper aims to address this particular issue and design a new type of scheme for hierarchical quantum communication, which we would refer to as hierarchical joint remote state preparation (HJRSP) scheme in analogy with the well known joint remote state preparation (JRSP) schemes [17, 18, 19, 20].

Here, it would be apt to note that there exists a variant of teleportation known as RSP, where a known quantum state is remotely prepared at the receiver's end. Thus, RSP may be viewed as teleportation of a known quantum state. The first RSP scheme was proposed by Pati [13] using Bell states, in 2000. This scheme required 1 bit of classical communication and a shared entangled state (1 ebit). As the standard teleportation scheme requires 2 cbit and 1 ebit, Pati's scheme of RSP was able to teleport a known quantum state with reduced resources compared to the unknown qubit case. However, this scheme was probabilistic in nature, as the success probability was not unity. Later a deterministic counterpart of the RSP scheme was proposed by An et al. [21], where the required resources become equal to that of teleportation [5]. Subsequently, a large amount of work has been carried out on RSP. In these works RSP has been implemented in probabilistic, deterministic, controlled and controlled bidirectional manner using different quantum states, such as n -level, 4-qubit GHZ state, multi-qubit GHZ state, 4-qubit cluster-type state, arbitrary two qubit state, and W state [17, 18, 19, 20, 22, 23, 24, 25, 26, 27]. Interestingly, a few experimental realizations of some of the RSP schemes have also been reported in the past [28, 29, 30, 31, 32, 33].

In 2008, the RSP scheme was modified to a three-party scheme of the joint RSP (JRSP) [17]. This one is a unique quantum communication scheme, where two senders jointly prepare a known quantum state at the remote port. The state to be prepared remotely is neither completely known to sender 1 nor to sender 2, but they jointly know the state to be teleported. It is worth stressing that this scheme has no analogue in teleportation, as an unknown quantum state cannot be teleported in this way by more than one sender. Since then several schemes for JRSP have been proposed [18, 19, 20]. In fact, the JRSP [34] scheme was investigated under the amplitude damping (AD) and phase damping (PD) noise models in the recent past. Interestingly, practical applications of the hierarchical quantum communication schemes, described in the context of HQIS [12] and HDQSS [16], motivated us to investigate the possibility of designing a hierarchical version of JRSP (HJRSP) scheme with the hope of using it in some of the practical purposes of daily life. Further, recent investigations on RSP schemes [22, 34] under the noisy environment, such as AD and PD channels, motivated us to simulate a similar study for the proposed HJRSP protocol.

The HQIS scheme [12] proposed in the recent past involves teleportation of an unknown quantum state among three receivers hierarchically. Further, we know that in a JRSP scheme, there are two senders having the information of amplitude and relative phase, respectively. Hence, the proposed HJRSP scheme should have at least five parties (2 senders and three receivers to ensure joint preparation and hierarchical reconstruction), and therefore, we need at least a 5-qubit quantum state for the implementation of the HJRSP protocol. This is why, we have used a 5-qubit

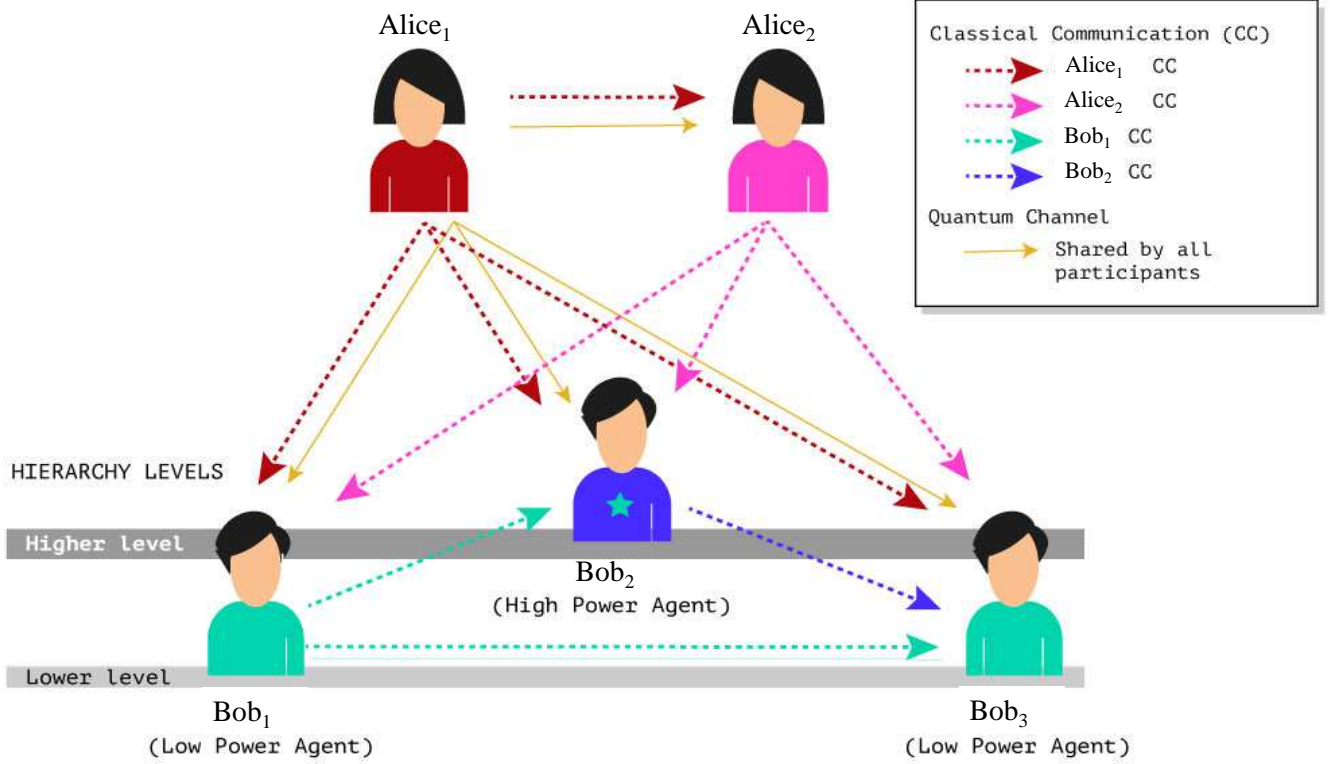


Figure 1: HJRSP scheme illustrated through a schematic diagram. The quantum and classical communications involved between all the parties are shown using the smooth and dashed lines, respectively. It is shown that Bob₂ can reconstruct the quantum state with the help of only one of the lower power agents (in the diagram it is shown as Bob₁). In contrast, Bob₃ requires help of both Bob₁ and Bob₂. Here, we have not shown the case when Bob₁ reconstructs the quantum state as it is similar to the case when Bob₃ does so

cluster state of the form

$$|C\rangle = \frac{1}{2} [|00000\rangle + |00111\rangle + |11010\rangle + |11101\rangle]_{S_1 S_2 R_1 R_2 R_3} \quad (1)$$

with the qubits S_1, S_2 belonging to the senders Alice₁ and Alice₂; and R_1, R_2 and R_3 corresponding to the qubits of the three receivers, who are referred to as Bob₁, Bob₂, and Bob₃, respectively. The preparation of the cluster state (1) by Alice₁ and distribution of qubits to all other parties is illustrated through the schematic diagram shown in Fig. 1. More detail of the scheme follows in the forthcoming section.

In the following sections, the paper is organized as follows. In Sec. 2, we propose a protocol of deterministic HJRSP using 5-qubit cluster state of the form (1). This is followed by a protocol of probabilistic HJRSP using a non-maximally entangled counterpart of the 5-qubit cluster state (1). Subsequently, we have discussed some practical applications of the proposed HJRSP schemes in Sec. 3. Further, in Sec. 4, we study the effect of a set of noise models on the proposed deterministic HJRSP scheme. Finally, we conclude the paper in Sec. 5.

2 Hierarchical joint remote state Preparation with 5-qubit cluster state

In a HJRSP scheme, we have two senders (Alice₁ and Alice₂), such that Alice₁ knows the information about the amplitude, and Alice₂ has the phase information related to the qubit $|\xi\rangle = a|0\rangle + be^{i\phi}|1\rangle : a^2 + b^2 = 1$. It implies that the senders jointly know the quantum state to be prepared at the remote location (i.e., at the receivers' end), whereas it is unknown to the receivers. Further, we require at least three receivers to design a hierarchical scheme for joint remote state preparation. Here, the three receivers are referred to as Bob₁, Bob₂ and Bob₃, respectively. The hierarchy among the receivers will be established in what follows. To accomplish the task, we have chosen a 5-qubit cluster state of the form (1) where each of the participants possesses one qubit of the 5-qubit cluster state.

Subsequently, a probabilistic version of HJRSP scheme is introduced, and it's shown that the probabilistic HJRSP scheme appears when we use a non-maximally entangled state of the form (1) as a quantum channel.

2.1 Deterministic HJRSP

We may now describe a protocol of deterministic HJRSP using a quantum state of the form (1) in following steps:

Step 1 Alice₁ prepares the 5-qubit cluster state (1). She keeps S_1 (1st) qubit with herself and sends S_2 (2nd) qubit to Alice₂ and R_1, R_2, R_3 qubits (3rd, 4th and 5th qubits) to the three receivers Bob₁, Bob₂, Bob₃, respectively.

Step 2 Alice₁ measures her qubit in $\{|u_0\rangle, |u_1\rangle\}$ basis, where $|u_0\rangle = a|0\rangle + b|1\rangle$ and $|u_1\rangle = b|0\rangle - a|1\rangle$, and announces the measurement outcome.

As the cluster state (1) can be expressed as

$$\begin{aligned} |C\rangle &= \frac{1}{2} [|u_0\rangle_{S_1} \{a(|0000\rangle + |0111\rangle) + b(|1010\rangle + |1101\rangle)\}_{S_2 R_1 R_2 R_3} \\ &+ |u_1\rangle_{S_1} \{b(|0000\rangle + |0111\rangle) - a(|1010\rangle + |1101\rangle)\}_{S_2 R_1 R_2 R_3}], \end{aligned} \quad (2)$$

the measurement of Alice₁ would reduce $|C\rangle$ to $|\Psi_0\rangle$ ($|\Psi_1\rangle$) if her measurement outcome is $|u_0\rangle$ ($|u_1\rangle$), where

$$\begin{aligned} |\Psi_0\rangle &= \frac{1}{\sqrt{2}} \{a(|0000\rangle + |0111\rangle) + b(|1010\rangle + |1101\rangle)\}_{S_2 R_1 R_2 R_3}, \\ |\Psi_1\rangle &= \frac{1}{\sqrt{2}} \{b(|0000\rangle + |0111\rangle) - a(|1010\rangle + |1101\rangle)\}_{S_2 R_1 R_2 R_3}. \end{aligned} \quad (3)$$

Case 1: In Step 2, Alice₁'s measurement yields $|u_0\rangle$

Step 3 If Alice₁'s measurement yields $|u_0\rangle$ then Alice₂ applies a single qubit phase gate $P(2\phi) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2i\phi} \end{pmatrix}$ on her qubit and that transforms $|\Psi_0\rangle$ to

$$\begin{aligned} |\Psi'_0\rangle &= (P(2\phi) \otimes I_2^{\otimes 3}) |\Psi_0\rangle = \frac{1}{\sqrt{2}} \{a(|0000\rangle + |0111\rangle) + be^{2i\phi}(|1010\rangle + |1101\rangle)\}_{S_2 R_1 R_2 R_3} \\ &= \frac{1}{2} [|v_0\rangle_{S_2} \{a(|000\rangle + |111\rangle) + be^{i\phi}(|010\rangle + |101\rangle)\}_{R_1 R_2 R_3} \\ &+ e^{i\phi} |v_1\rangle_{S_2} \{a(|000\rangle + |111\rangle) - be^{i\phi}(|010\rangle + |101\rangle)\}_{R_1 R_2 R_3}] \\ &= \frac{1}{\sqrt{2}} (|v_0\rangle_{S_2} |\phi_0\rangle_{R_1 R_2 R_3} + |v_1\rangle_{S_2} |\phi_1\rangle_{R_1 R_2 R_3}), \end{aligned} \quad (4)$$

where $|v_0\rangle = \frac{|0\rangle + e^{i\phi}|1\rangle}{\sqrt{2}}$, $|v_1\rangle = \frac{e^{-i\phi}|0\rangle - |1\rangle}{\sqrt{2}}$, $|\phi_0\rangle = \frac{1}{\sqrt{2}} \{a(|000\rangle + |111\rangle) + be^{i\phi}(|010\rangle + |101\rangle)\}_{R_1 R_2 R_3}$, and $|\phi_1\rangle = \frac{1}{\sqrt{2}} \{a(|000\rangle + |111\rangle) - be^{i\phi}(|010\rangle + |101\rangle)\}_{R_1 R_2 R_3}$.

Step 4 Alice₂ measures her qubit S_2 in $\{|v_0\rangle, |v_1\rangle\}$ basis and announces the outcome obtained by her.

From Eq. (4) we can easily observe that if Alice₂ obtains v_0 (v_1) then the composite state shared by the receivers reduces to $|\phi_0\rangle$ ($|\phi_1\rangle$).

Case 1.1: The receivers decide that Bob₂ will reconstruct the state

Step 5 If the agents decide that Bob₂ will reconstruct the unknown quantum state $|\xi\rangle$, then $|\phi_0\rangle$ and $|\phi_1\rangle$ can be decomposed as

$$\begin{aligned} |\phi_0\rangle &= \frac{1}{\sqrt{2}} \{ |00\rangle_{R_1 R_3} (a|0\rangle + be^{i\phi}|1\rangle)_{R_2} + |11\rangle_{R_1 R_3} (a|1\rangle + be^{i\phi}|0\rangle)_{R_2} \}, \\ |\phi_1\rangle &= \frac{1}{\sqrt{2}} \{ |00\rangle_{R_1 R_3} (a|0\rangle - be^{i\phi}|1\rangle)_{R_2} + |11\rangle_{R_1 R_3} (a|1\rangle - be^{i\phi}|0\rangle)_{R_2} \}. \end{aligned} \quad (5)$$

From Eq. (5), it is clear that Bob₂ can reconstruct the quantum state $|\xi\rangle$ with the collaboration of either Bob₁ or Bob₃ by applying Pauli operations I , X , Z and iY , respectively, as shown in Table 1. It is important to note that Bob₂ needs the collaboration of either the receiver Bob₁ or Bob₃. He does not require collaboration of both as the measurement performed by the receivers Bob₁ and Bob₃ in computational basis (i.e., $\{|0\rangle, |1\rangle\}$) always yield the same outcome (cf. Column 3 of Table 1). Thus, the communication of the measurement outcome from anyone of the receivers Bob₁ or Bob₃ is sufficient for the receiver Bob₂. Consequently, the receiver Bob₂ is the higher power agent in the proposed HJRSP scheme.

Alice ₁ 's measurement outcome in $\{u_0, u_1\}$ basis	Alice ₂ 's measurement outcome in $\{v_0, v_1\}$ basis	Bob _i 's measurement outcome (in $\{ 0\rangle, 1\rangle\}$ basis), where $i \in \{1, 3\}$	Pauli operation to be applied by Bob ₂
$ u_0\rangle$	$ v_0\rangle$	$ 0\rangle$	I
		$ 1\rangle$	X
	$ v_1\rangle$	$ 0\rangle$	Z
		$ 1\rangle$	iY

Table 1: The relation between the measurement outcomes of the senders and receivers with the unitary operations to be applied by the receiver Bob₂ to recover the quantum state jointly sent by the senders Alice₁ and Alice₂. Bob₂ needs the collaboration of either of the agents Bob₁ or Bob₃.

Case 1.2: The receivers decide that Bob₃ will reconstruct the state

Step 5' In case the agents wish Bob₃ to reconstruct $|\xi\rangle$, then $|\phi_0\rangle$ and $|\phi_1\rangle$ can be decomposed as

$$\begin{aligned}
|\phi_0\rangle &= \frac{1}{2} \left[(|+\rangle|0\rangle)_{R_1 R_2} (a|0\rangle + be^{i\phi}|1\rangle)_{R_3} + (|-\rangle|0\rangle)_{R_1 R_2} (a|0\rangle - be^{i\phi}|1\rangle)_{R_3} \right. \\
&\quad \left. + (|+\rangle|1\rangle)_{R_1 R_2} (a|1\rangle + be^{i\phi}|0\rangle)_{R_3} - (|-\rangle|1\rangle)_{R_1 R_2} (a|1\rangle - be^{i\phi}|0\rangle)_{R_3} \right], \\
|\phi_1\rangle &= \frac{1}{2} \left[(|+\rangle|0\rangle)_{R_1 R_2} (a|0\rangle - be^{i\phi}|1\rangle)_{R_3} + (|-\rangle|0\rangle)_{R_1 R_2} (a|0\rangle + be^{i\phi}|1\rangle)_{R_3} \right. \\
&\quad \left. + (|+\rangle|1\rangle)_{R_1 R_2} (a|1\rangle - be^{i\phi}|0\rangle)_{R_3} - (|-\rangle|1\rangle)_{R_1 R_2} (a|1\rangle + be^{i\phi}|0\rangle)_{R_3} \right].
\end{aligned} \tag{6}$$

From Eq. (6), it is clear that Bob₃ can reconstruct the quantum state $|\xi\rangle$ by applying the Pauli operations as shown in Table 2 iff both the receivers Bob₁ and Bob₂ cooperate simultaneously by measuring their qubits (in $\{|+\rangle, |-\rangle\}$ basis and computational basis, respectively) and sharing the measurement outcomes. This fact can be established mathematically by tracing over R_1 and R_2 qubits, which gives a completely mixed state for the receiver Bob₃. The measurement outcomes of the receivers Bob₁ and Bob₂ are summarized in Column 3 of Table 2 with the corresponding Pauli operations Bob₃ has to apply in the next column. Here, it is important to note that Bob₃ needs the collaboration of both Bob₁ and Bob₂, whereas in Case 1.1, we have seen that Bob₂ can reconstruct the quantum state with the help of either Bob₁ or Bob₃. Thus, Bob₂ is a higher power agent. A similar analysis would reveal that Bob₁ is also a lower power agent as to reconstruct the unknown quantum state $|\xi\rangle$ sent by the senders he will also require the help of the remaining two Bobs.

Alice ₁ 's measurement outcome in $\{u_0, u_1\}$ basis	Alice ₂ 's measurement outcome in $\{v_0, v_1\}$ basis	Bob ₁ 's and Bob ₂ 's joint measurement outcome (in $\{ +\rangle, -\rangle\}$ and $\{ 0\rangle, 1\rangle\}$ bases, respectively)	Pauli operation to be applied by Bob ₃
$ u_0\rangle$	$ v_0\rangle$	$ +\rangle 0\rangle$	I
		$ -\rangle 0\rangle$	Z
		$ +\rangle 1\rangle$	X
		$ -\rangle 1\rangle$	iY
	$ v_1\rangle$	$ +\rangle 0\rangle$	Z
		$ -\rangle 0\rangle$	I
		$ +\rangle 1\rangle$	iY
		$ -\rangle 1\rangle$	X

Table 2: The measurement outcomes of all the senders and receivers, and corresponding Pauli operations Bob₃ has to apply to recover the quantum state. The receiver Bob₃ needs the joint collaboration of all other agents.

Case 2: In Step2, Alice₁'s measurement yields $|u_1\rangle$

If Alice₁'s measurement yields $|u_1\rangle$, then **Step3** and the following steps described above will be modified to **Step3-1** and so on as described in the following steps:

Step 3-1 If Alice₁'s measurement outcome is $|u_1\rangle$ then Alice₂ need not apply the phase gate $P(2\phi)$, and the state can be written as

$$\begin{aligned} |\Psi_1\rangle &= \frac{1}{\sqrt{2}} [b(|0000\rangle + |0111\rangle) - a(|1010\rangle + |1101\rangle)]_{S_2 R_1 R_2 R_3} \\ &= \frac{1}{2} [|v_0\rangle_{S_2} \{b(|000\rangle + |111\rangle) - ae^{-i\phi}(|010\rangle + |101\rangle)\}_{R_1 R_2 R_3} \\ &\quad + |v_1\rangle_{S_2} \{be^{i\phi}(|000\rangle + |111\rangle) + a(|010\rangle + |101\rangle)\}_{R_1 R_2 R_3}] \\ &= \frac{1}{\sqrt{2}} [|v_0\rangle_{S_2} |\phi'_0\rangle_{R_1 R_2 R_3} + |v_1\rangle_{S_2} |\phi'_1\rangle_{R_1 R_2 R_3}]. \end{aligned} \quad (7)$$

Subsequently, Alice₂ measures her qubit S_2 in $\{v_0, v_1\}$ basis, where basis elements are already described in Step 3. The reduced state at the end of the three receivers can be deduced from Eq. (7).

Case 2.1: The receivers decide that Bob₂ will reconstruct the state

Step 4-1 When Bob₂ is supposed to reconstruct the unknown quantum state $|\xi\rangle$, the reduced quantum state can be decomposed as

$$\begin{aligned} |\phi'_0\rangle &= \frac{1}{\sqrt{2}} e^{-i\phi} \{ |00\rangle_{R_1 R_3} (be^{i\phi}|0\rangle - a|1\rangle)_{R_2} + |11\rangle_{R_1 R_3} (be^{i\phi}|1\rangle - a|0\rangle)_{R_2} \}, \\ |\phi'_1\rangle &= \frac{1}{\sqrt{2}} \{ |00\rangle_{R_1 R_3} (be^{i\phi}|0\rangle + a|1\rangle)_{R_2} + |11\rangle_{R_1 R_3} (be^{i\phi}|1\rangle + a|0\rangle)_{R_2} \}, \end{aligned} \quad (8)$$

where the global phase $e^{-i\phi}$ can be ignored, which is consistent with the quantum mechanics.

From Eq. (8), it is clear that Bob₂ can reconstruct the quantum state $|\xi\rangle$ by applying suitable Pauli operations with the help of measurement results of either Bob₁ or Bob₃ as both of them have the same measurement outcomes (cf. Column 3 of Table 3). Consequently, the receiver R_2 is accredited the higher power agent in the proposed hierarchical scheme.

Alice ₁ 's measurement outcome in $\{u_0, u_1\}$ basis	Alice ₂ 's measurement outcome in $\{v_0, v_1\}$ basis	Bob _i 's measurement outcome (in $\{ 0\rangle, 1\rangle\}$ basis), where $i \in \{1, 3\}$	Pauli operations Bob ₂ have to apply
$ u_1\rangle$	$ v_0\rangle$	$ 0\rangle$	iY
		$ 1\rangle$	Z
	$ v_1\rangle$	$ 0\rangle$	X
		$ 1\rangle$	I

Table 3: When Alice₁'s measurement outcome is $|u_1\rangle$, corresponding measurement results of all the remaining parties and the unitary operations to be applied by Bob₂ are summarized here. The receiver Bob₂ needs the collaboration of either of the agents Bob₁ or Bob₃.

Case 2.2: The receivers decide that Bob₃ will reconstruct the state

Step 4 – 1' When Bob₃ has to reconstruct the quantum state, the reduced quantum state can be written (corresponding to measurement outcomes of Alice₂) as follows:

If the measurement outcome of Alice₂ is $|v_0\rangle$, the reduced state becomes

$$\begin{aligned} |\phi'_0\rangle &= \frac{e^{-i\phi}}{2} \left[(|+\rangle|0\rangle)_{R_1 R_2} (be^{i\phi}|0\rangle - a|1\rangle)_{R_3} + (|-\rangle|0\rangle)_{R_1 R_2} (be^{i\phi}|0\rangle + a|1\rangle)_{R_3} \right. \\ &\quad \left. + (|+\rangle|1\rangle)_{R_1 R_2} (be^{i\phi}|1\rangle - a|0\rangle)_{R_3} - (|-\rangle|1\rangle)_{R_1 R_2} (be^{i\phi}|1\rangle + a|0\rangle)_{R_3} \right], \end{aligned} \quad (9)$$

and if the measurement outcome of the sender Alice₂ is $|v_1\rangle$, the state can be written as

$$\begin{aligned} |\phi'_1\rangle &= \frac{1}{2} \left[(|+\rangle|0\rangle)_{R_1 R_2} (be^{i\phi}|0\rangle + a|1\rangle)_{R_3} + (|-\rangle|0\rangle)_{R_1 R_2} (be^{i\phi}|0\rangle - a|1\rangle)_{R_3} \right. \\ &\quad \left. + (|+\rangle|1\rangle)_{R_1 R_2} (be^{i\phi}|1\rangle + a|0\rangle)_{R_3} - (|-\rangle|1\rangle)_{R_1 R_2} (be^{i\phi}|1\rangle - a|0\rangle)_{R_3} \right]. \end{aligned} \quad (10)$$

From Eqs. (9) and (10), it can be inferred that Bob₃ can reconstruct $|\xi\rangle$ by applying an appropriate Pauli operation only if both the remaining receivers help him by conveying their measurement results after measuring their qubits in suitable basis. In Table 4, it has been explicitly shown that Bob₁ and Bob₂ measure their qubits in $\{|+\rangle, |-\rangle\}$ and $\{|0\rangle, |1\rangle\}$ bases, respectively. In the table, Pauli operations corresponding to each possible measurement outcome are also clearly mentioned. Hence, Bob₃ is the lower power agent in the proposed HJRSP scheme. Interestingly, a similar situation is obtained for Bob₁, consequently he is at the same level of hierarchy as Bob₃.

Alice ₁ 's measurement outcome in { u_0, u_1 } basis	Alice ₂ 's measurement outcome in { v_0, v_1 } basis	Bob ₁ 's and Bob ₂ 's joint measurement outcome (in { $ +\rangle, -\rangle$ }, { $ 0\rangle, 1\rangle$ } bases, respectively)	Pauli operation to be applied by Bob ₃
$ u_1\rangle$	$ v_0\rangle$	$ +\rangle 0\rangle$	iY
		$ -\rangle 0\rangle$	X
		$ +\rangle 1\rangle$	Z
		$ -\rangle 1\rangle$	I
	$ v_1\rangle$	$ +\rangle 0\rangle$	X
		$ -\rangle 0\rangle$	iY
		$ +\rangle 1\rangle$	I
		$ -\rangle 1\rangle$	Z

Table 4: The measurement outcomes of both the senders and receivers are summarized with the Pauli operations that Bob₃ needs to apply. The receiver Bob₃ needs the joint collaboration of all other agents.

2.2 Probabilistic HJRSP

In the last subsection, we have proposed a deterministic HJRSP scheme using maximally entangled 5-qubit cluster state. In this section, we aim to propose a scheme for probabilistic HJRSP using non-maximally entangled 5-qubit cluster state of a specific form. To do so, we assume that Alice₁ prepares and shares a non-maximally entangled 5-qubit cluster state of the form

$$\begin{aligned}
|C'\rangle &= \frac{1}{\sqrt{2}} [\alpha(|00000\rangle + |00111\rangle) + \beta(|11010\rangle + |11101\rangle)]_{S_1 S_2 R_1 R_2 R_3} \\
&= \frac{1}{\sqrt{2}} [|u_0\rangle_{S_1} \{\alpha a(|0000\rangle + |0111\rangle) + \beta b(|1010\rangle + |1101\rangle)\} \\
&\quad + |u_1\rangle_{S_1} \{\alpha b(|0000\rangle + |0111\rangle) - \beta a(|1010\rangle + |1101\rangle)\}]_{S_2 R_1 R_2 R_3},
\end{aligned} \tag{11}$$

where $|\alpha|^2 + |\beta|^2 = 1$ with $|\alpha| \neq \frac{1}{\sqrt{2}}$ and $|u_0\rangle$ and $|u_1\rangle$ are already defined in the context of the previous protocol. From Eq. (11) it can be observed that if Alice₁ measures her qubit S_1 in $\{|u_0\rangle, |u_1\rangle\}$ basis, then the reduced quantum state corresponding to measurement outcome $|u_0\rangle$ and $|u_1\rangle$ are given as

$$\begin{aligned}
|\psi_0\rangle &= \frac{1}{\sqrt{2(a^2|\alpha|^2 + b^2|\beta|^2)}} [\alpha a(|0000\rangle + |0111\rangle) + \beta b(|1010\rangle + |1101\rangle)]_{S_2 R_1 R_2 R_3}, \\
|\psi_1\rangle &= \frac{1}{\sqrt{2(b^2|\alpha|^2 + a^2|\beta|^2)}} [\alpha b(|0000\rangle + |0111\rangle) - \beta a(|1010\rangle + |1101\rangle)]_{S_2 R_1 R_2 R_3},
\end{aligned} \tag{12}$$

respectively.

Case 1: Alice₁'s measurement outcome is $|u_0\rangle$

Then Alice₂ applies a phase operator $P(2\phi) = \begin{bmatrix} 1 & 0 \\ 0 & e^{2i\phi} \end{bmatrix}$ on $|\psi_0\rangle$, as in the deterministic HJRSP scheme defined above, and the state can be written as

$$|\psi'_0\rangle = (P(2\phi) \otimes I^{\otimes 3}) |\psi_0\rangle = \frac{1}{\sqrt{2(a^2|\alpha|^2 + b^2|\beta|^2)}} [\alpha a(|0000\rangle + |0111\rangle) + \beta b e^{2i\phi} (|1010\rangle + |1101\rangle)]_{S_2 R_1 R_2 R_3}. \tag{13}$$

As the transformed state can be written as

$$\begin{aligned}
|\psi'_0\rangle &= \frac{1}{2\sqrt{(a^2|\alpha|^2 + b^2|\beta|^2)}} [|v_0\rangle_{S_2} \{\alpha a(|000\rangle + |111\rangle) + \beta b e^{i\phi} (|010\rangle + |101\rangle)\} \\
&\quad + e^{i\phi} |v_1\rangle_{S_2} \{\alpha a(|000\rangle + |111\rangle) - \beta b e^{i\phi} (|010\rangle + |101\rangle)\}]_{R_1 R_2 R_3} \\
&= \frac{1}{\sqrt{2}} [|v_0\rangle_{S_2} |\Phi_0\rangle_{R_1 R_2 R_3} + |v_1\rangle_{S_2} |\Phi_1\rangle_{R_1 R_2 R_3}],
\end{aligned} \tag{14}$$

where $|\Phi_0\rangle = \frac{1}{\sqrt{2(a^2|\alpha|^2 + b^2|\beta|^2)}} \{\alpha a(|000\rangle + |111\rangle) + \beta b e^{i\phi} (|010\rangle + |101\rangle)\}_{R_1 R_2 R_3}$ and $|\Phi_1\rangle = \frac{1}{\sqrt{2(a^2|\alpha|^2 + b^2|\beta|^2)}} \{\alpha a(|000\rangle + |111\rangle) - \beta b e^{i\phi} (|010\rangle + |101\rangle)\}_{R_1 R_2 R_3}$. Further, Alice₂ measures her qubit S_2 in $\{v_0, v_1\}$ basis, where $|v_0\rangle$ and $|v_1\rangle$ are as defined in the previous subsection.

Case 1.1: The receivers decide that Bob₂ will recover the state

If the agents decide that Bob₂ will reconstruct the unknown quantum state $|\xi\rangle$, and other two receivers would measure their qubits in computational basis, then the reduced quantum state can be decomposed as

$$\begin{aligned} |\Phi_0\rangle &= \frac{1}{\sqrt{2(a^2|\alpha|^2+b^2|\beta|^2)}}\{|00\rangle_{R_1 R_3}(\alpha a|0\rangle + \beta b e^{i\phi}|1\rangle)_{R_2} + |11\rangle_{R_1 R_3}(\alpha a|1\rangle + \beta b e^{i\phi}|0\rangle)_{R_2}\}, \\ |\Phi_1\rangle &= \frac{1}{\sqrt{2(a^2|\alpha|^2+b^2|\beta|^2)}}\{|00\rangle_{R_1 R_3}(\alpha a|0\rangle - \beta b e^{i\phi}|1\rangle)_{R_2} + |11\rangle_{R_1 R_3}(\alpha a|1\rangle - \beta b e^{i\phi}|0\rangle)_{R_2}\}. \end{aligned} \quad (15)$$

From Eq. (15), it is clear that Bob₂ can not directly reconstruct the quantum state just by applying the Pauli operators even if all the receivers cooperate. Therefore, he has to change the strategy as follows: Bob₂ prepares an ancilla qubit in $|0\rangle_{\text{aux}}$ and applies the following 2-qubit unitary operations U_0/U_1 on his qubits (i.e., on the combined system of his existing qubit and ancilla), where

$$U_0 = \begin{pmatrix} \frac{\beta}{\alpha} & \sqrt{1 - \frac{\beta^2}{\alpha^2}} & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ \sqrt{1 - \frac{\beta^2}{\alpha^2}} & -\frac{\beta}{\alpha} & 0 & 0 \end{pmatrix}, \quad (16)$$

and

$$U_1 = U_0 (X \otimes I) = \begin{pmatrix} 0 & 0 & \frac{\beta}{\alpha} & \sqrt{1 - \frac{\beta^2}{\alpha^2}} \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1 - \frac{\beta^2}{\alpha^2}} & -\frac{\beta}{\alpha} \end{pmatrix}. \quad (17)$$

As α and β are known, the construction of U_0/U_1 is possible. In Eq. (15), let us consider a case that the measurement outcomes of Alice₂ and Bob₁ (or Bob₃ as the measurement outcomes of both lower power agents are the same) are $|v_0\rangle$ and $|0\rangle(|1\rangle)$, respectively. Then Bob₂ has to apply $U_0(U_1)$ on his product state. For instance,

$$|\chi\rangle_{1'} = |\chi\rangle_1 |0\rangle_{\text{aux}} = \frac{1}{\sqrt{(a^2|\alpha|^2 + b^2|\beta|^2)}} (\alpha a|0\rangle + \beta b e^{i\phi}|1\rangle) |0\rangle_{\text{aux}},$$

where $|\chi\rangle_1$ is Bob₂'s reduced quantum state corresponding to Bob₁'s measurement outcome $|0\rangle$. Subsequent operation of U_0 on this composite state gives

$$U_0|\chi\rangle_{1'} = \frac{1}{\sqrt{(a^2|\alpha|^2 + b^2|\beta|^2)}} \left\{ \beta(a|0\rangle + b e^{i\phi}|1\rangle)|0\rangle + a\sqrt{\alpha^2 - \beta^2}|1\rangle|1\rangle \right\}.$$

Finally, Bob₂ measures the last qubit (ancilla) in the computational basis $\{|0\rangle, |1\rangle\}$. If his measurement yields $|0\rangle$ then he obtains unknown state with unit fidelity, but if his measurement on ancilla yields $|1\rangle$ then he fails to reconstruct the state. In other words, he can recover the unknown state only when he obtains a specific measurement outcome. Thus, in analogy to the probabilistic teleportation scheme we may refer to this scheme as probabilistic hierarchical joint remote state preparation scheme. Similarly, we can check the other three possibilities. All these results are summarized in the Table 5.

Alice ₁ 's measure- ment outcome in $\{u_0, u_1\}$ basis	Alice ₂ 's measure- ment outcome in $\{v_0, v_1\}$ basis	Bob _i 's measurement outcome (in $\{ 0\rangle, 1\rangle\}$ basis), where $i \in \{1, 3\}$	Two qubit unitary operation to be applied by Bob ₂ (U_0/U_1)	Pauli operation Bob ₂ have to apply to obtain $ \xi\rangle$
$ u_0\rangle$	$ v_0\rangle$	$ 0\rangle$	U_0	I
		$ 1\rangle$	U_1	I
	$ v_1\rangle$	$ 0\rangle$	U_0	Z
		$ 1\rangle$	U_1	Z

Table 5: The probabilistic HJRSP scheme is summarized here for all the successful cases. The measurement outcomes of all the senders and receivers are summed up with the unitary and Pauli operations Bob₂ needs to apply. It can be deduced from the third column that the receiver Bob₂ is a higher power agent as he needs the collaboration of either of the agents.

Case 1.2: The receivers decide that Bob₃ will recover the state

If the agents unanimously agreed that Bob₃ will reconstruct the unknown quantum state, the reduced quantum state of three receivers can be decomposed as follows.

When the measurement outcome of Alice₂ is $|v_0\rangle$

$$|\Phi_0\rangle = \frac{1}{2\sqrt{(a^2|\alpha|^2+b^2|\beta|^2)}} \left[(|+\rangle|0\rangle)_{R_1 R_2} (\alpha a|0\rangle + \beta b e^{i\phi}|1\rangle)_{R_3} + (|-\rangle|0\rangle)_{R_1 R_2} (\alpha a|0\rangle - \beta b e^{i\phi}|1\rangle)_{R_3} \right. \\ \left. + (|+\rangle|1\rangle)_{R_1 R_2} (\alpha a|1\rangle + \beta b e^{i\phi}|0\rangle)_{R_3} - (|-\rangle|1\rangle)_{R_1 R_2} (\alpha a|1\rangle - \beta b e^{i\phi}|0\rangle)_{R_3} \right], \quad (18)$$

When the measurement outcome of Alice₂ is $|v_1\rangle$

$$|\Phi_1\rangle = \frac{1}{2\sqrt{(a^2|\alpha|^2+b^2|\beta|^2)}} \left[(|+\rangle|0\rangle)_{R_1 R_2} (\alpha a|0\rangle - \beta b e^{i\phi}|1\rangle)_{R_3} + (|-\rangle|0\rangle)_{R_1 R_2} (\alpha a|0\rangle + \beta b e^{i\phi}|1\rangle)_{R_3} \right. \\ \left. + (|+\rangle|1\rangle)_{R_1 R_2} (\alpha a|1\rangle - \beta b e^{i\phi}|0\rangle)_{R_3} - (|-\rangle|1\rangle)_{R_1 R_2} (\alpha a|1\rangle + \beta b e^{i\phi}|0\rangle)_{R_3} \right]. \quad (19)$$

From Eqs. (18) and (19), it is clear that Bob₃ can not directly reconstruct the quantum state as in the previous case. Hence, similar to the strategy followed by Bob₂ above, Bob₃ also applies a unitary operation U_0/U_1 on his composite system (i.e., on his existing qubit and ancilla). Finally, he can reconstruct the quantum state by further application of an appropriate Pauli operation as shown in Table 6.

Alice ₁ 's measurement outcome in $\{u_0, u_1\}$ basis	Alice ₂ 's measurement outcome in $\{v_0, v_1\}$ basis	Bob ₁ 's measurement outcome (in $\{ +\rangle, -\rangle\}$ basis)	Bob ₂ 's measurement outcomes (in $\{ 0\rangle, 1\rangle\}$ basis)	Two qubit unitary operation to be applied by Bob ₃ (U_0/U_1)	Pauli operation to be applied by Bob ₃ to recon- struct $ \xi\rangle$
$ u_0\rangle$	$ v_0\rangle$	$ +\rangle$	$ 0\rangle$	U_0	I
		$ -\rangle$	$ 0\rangle$	U_0	Z
		$ +\rangle$	$ 1\rangle$	U_1	I
		$ -\rangle$	$ 1\rangle$	U_1	Z
	$ v_1\rangle$	$ +\rangle$	$ 0\rangle$	U_0	Z
		$ -\rangle$	$ 0\rangle$	U_0	I
		$ +\rangle$	$ 1\rangle$	U_1	Z
		$ -\rangle$	$ 1\rangle$	U_1	I

Table 6: All possible successful cases in the probabilistic HJRSP scheme are summarized with the measurement outcomes of all the senders and receivers with corresponding unitary and Pauli operations to be implemented by Bob₃ to recover the quantum state. The receiver Bob₃ needs the joint collaboration of all the other agents.

Case 2: Alice₁'s measurement outcome is $|u_1\rangle$

Then Alice₂ need not apply a phase operator $P(2\phi)$ on $|\psi_1\rangle$ in Eq. (12). Subsequently, Alice₂ measures her qubit S_2 in $\{v_0, v_1\}$ basis, where $|v_0\rangle$ and $|v_1\rangle$ have the same meaning as in the last subsection. As the state can be written as

$$|\psi''\rangle = \frac{1}{2\sqrt{(b^2|\alpha|^2+a^2|\beta|^2)}} \left[|v_0\rangle_{S_2} \{ \alpha b(|000\rangle + |111\rangle) - \beta a e^{-i\phi}(|010\rangle + |101\rangle) \}_{R_1 R_2 R_3} \right. \\ \left. + |v_1\rangle_{S_2} \{ e^{i\phi} \alpha b(|000\rangle + |111\rangle) + \beta a(|010\rangle + |101\rangle) \}_{R_1 R_2 R_3} \right], \quad (20)$$

$$= \frac{1}{\sqrt{2}} \left[|v_0\rangle_{S_2} |\Phi'_0\rangle_{R_1 R_2 R_3} + |v_1\rangle_{S_2} |\Phi'_1\rangle_{R_1 R_2 R_3} \right],$$

from which the quantum state after measurement can be deduced to be $|\Phi'_0\rangle$ ($|\Phi'_1\rangle$) for Alice₂'s measurement outcomes $|v_0\rangle$ ($|v_1\rangle$).

Case 2.1: The receivers decide that Bob₂ will recover the state

If all the agents decree Bob₂ to reconstruct the unknown quantum state, the reduced quantum state of the receivers can be decomposed as

$$\begin{aligned} |\Phi'_0\rangle &= \frac{1}{\sqrt{2(b^2|\alpha|^2+a^2|\beta|^2)}} \{ |00\rangle_{R_1 R_3} (\alpha b|0\rangle - \beta a e^{-i\phi}|1\rangle)_{R_2} + |11\rangle_{R_1 R_3} (\alpha b|1\rangle - \beta a e^{-i\phi}|0\rangle)_{R_2} \}, \\ |\Phi'_1\rangle &= \frac{1}{\sqrt{2(b^2|\alpha|^2+a^2|\beta|^2)}} \{ |00\rangle_{R_1 R_3} (\alpha b e^{i\phi}|0\rangle + \beta a|1\rangle)_{R_2} + |11\rangle_{R_1 R_3} (\alpha b e^{i\phi}|1\rangle + \beta a|0\rangle)_{R_2} \}. \end{aligned} \quad (21)$$

From Eq. (21), it is concluded that Bob₂ can not directly reconstruct the quantum state unless he adopts the same strategy as was discussed for the case of Alice₁'s measurement outcome $|u_0\rangle$. In analogy of Case 1, when Alice₁ obtained $|u_0\rangle$, specific cases may be studied. Here, we have restricted ourselves only to the successful cases of probabilistic HJRSP which summarized in Table 7.

Alice ₁ 's measure- ment outcome in $\{u_0, u_1\}$ basis	Alice ₂ 's measure- ment outcome in $\{v_0, v_1\}$ basis	Bob _i 's measurement outcome (in $\{ 0\rangle, 1\rangle\}$ basis), where $i \in \{1, 3\}$	Two qubit unitary operation to be applied by Bob ₂ (U_0/U_1)	Pauli operation Bob ₂ have to apply to reconstruct $ \xi\rangle$
$ u_1\rangle$	$ v_0\rangle$	$ 0\rangle$	U_0	iY
		$ 1\rangle$	U_1	iY
	$ v_1\rangle$	$ 0\rangle$	U_0	X
		$ 1\rangle$	U_1	X

Table 7: All successful cases of probabilistic HJRSP are summarized with the measurement outcomes of all the senders and receivers with corresponding unitary and Pauli operations which Bob₂ applies to recover the quantum state. The receiver Bob₂ remains a higher power agent as in deterministic HJRSP scheme.

Case 2.2: The receivers decide that Bob₃ will recover the state

When Bob₃ is assigned the task to reconstruct the unknown quantum state, the reduced quantum state can be decomposed as follows.

When the measurement outcome of Alice₂ is $|v_0\rangle$

$$\begin{aligned} |\Phi'_0\rangle &= \frac{1}{2\sqrt{(b^2|\alpha|^2+a^2|\beta|^2)}} \left[(|+\rangle|0\rangle)_{R_1 R_2} (\alpha b|0\rangle - \beta a e^{-i\phi}|1\rangle)_{R_3} + (|-\rangle|0\rangle)_{R_1 R_2} (\alpha b|0\rangle + \beta a e^{-i\phi}|1\rangle)_{R_3} \right. \\ &\quad \left. + (|+\rangle|1\rangle)_{R_1 R_2} (\alpha b|1\rangle - \beta a e^{-i\phi}|0\rangle)_{R_3} - (|-\rangle|1\rangle)_{R_1 R_2} (\alpha b|1\rangle + \beta a e^{-i\phi}|0\rangle)_{R_3} \right], \end{aligned} \quad (22)$$

When Alice₂ obtains $|v_1\rangle$

$$\begin{aligned} |\Phi'_1\rangle &= \frac{1}{2\sqrt{(b^2|\alpha|^2+a^2|\beta|^2)}} \left[(|+\rangle|0\rangle)_{R_1 R_2} (\alpha b e^{i\phi}|0\rangle + \beta a|1\rangle)_{R_3} + (|-\rangle|0\rangle)_{R_1 R_2} (\alpha b e^{i\phi}|0\rangle - \beta a|1\rangle)_{R_3} \right. \\ &\quad \left. + (|+\rangle|1\rangle)_{R_1 R_2} (\alpha b e^{i\phi}|1\rangle + \beta a|0\rangle)_{R_3} - (|-\rangle|1\rangle)_{R_1 R_2} (\alpha b e^{i\phi}|1\rangle - \beta a|0\rangle)_{R_3} \right]. \end{aligned} \quad (23)$$

Similar to the earlier cases of probabilistic HJRSP, Bob₃ will have to apply one of the unitary operations U_0 or U_1 before reconstructing the state by operating a suitable Pauli gate, which depends on the measurement outputs of Bob₁ and Bob₂. All the successful cases of probabilistic HJRSP are entabulated in Table 8.

Alice ₁ 's measure- ment outcome in $\{u_0, u_1\}$ basis	Alice ₂ 's measure- ment outcome in $\{v_0, v_1\}$ basis	Bob ₁ 's measurement outcome (in $\{ +\rangle, -\rangle\}$ basis)	Bob ₂ 's measurement outcome (in $\{ 0\rangle, 1\rangle\}$ basis)	Two qubit unitary operation to be applied by Bob ₃ (U_0/U_1)	Pauli operation to be applied by Bob ₃ to recon- struct $ \xi\rangle$
$ u_1\rangle$	$ v_0\rangle$	$ +\rangle$	$ 0\rangle$	U_0	iY
		$ -\rangle$	$ 0\rangle$	U_0	X
		$ +\rangle$	$ 1\rangle$	U_1	iY
		$ -\rangle$	$ 1\rangle$	U_1	X
	$ v_1\rangle$	$ +\rangle$	$ 0\rangle$	U_0	X
		$ -\rangle$	$ 0\rangle$	U_0	iY
		$ +\rangle$	$ 1\rangle$	U_1	X
		$ -\rangle$	$ 1\rangle$	U_1	iY

Table 8: All the successful cases of Probabilistic HJRSP scheme when Alice₁'s measurement result is $|u_1\rangle$ are listed here. The unitary operation and suitable Pauli gates Bob₃ will require are mentioned against all possible measurement outcomes of the senders and receivers. Bob₃ can be seen the lowest power agent here as well.

3 Practical Applications

Hierarchical communications play important roles in our day to day life. Several examples of practical situations where hierarchical communication is essential have been discussed in our earlier works [12, 16]. Specifically, its relevance is discussed in context of HQSS [12], HQIS [11], and HDQSS [16]. In all those cases, the information or quantum state to be shared was in possession of a single person (whom we referred to as Alice or Sender). In contrast, here the initial state is jointly possessed by two senders. In many practical scenarios, it allows an avenue for joint decision, and thus, reduces risk associated with policy decisions taken by a single person. Let us elaborate this particular feature through a specific example.

Consider that there exists a code to unlock a nuclear weapon. Because of the fatal effect it may cause, this particular code cannot be given to a single authorized person. Thus, the information (code) has to be distributed among at least two authorized persons, so that none of the authorized persons can misuse the code. Now, consider that Alice₁ and Alice₂ (two authorized persons) are the Prime Minister and President of a country, and Bob₂ is the defense minister, Bob₁ and Bob₃ are the defense secretary and the chief of the armed forces of that country, respectively. If and only if the Prime Minister and President together wish to permit the use of the nuclear weapon at a suitable time, then they jointly distribute the information (the code which is required to unlock the nuclear weapon) among the defense minister, the defense secretary and the chief of the armed forces in such a way that the minister can unlock the weapon if either the defense secretary or the chief of the armed forces agrees and cooperates with him. However, if the chief of the armed forces or the defense secretary wants to unlock the weapon they would require the cooperation of each other and that of the defense minister. Thus, the defense minister is more powerful than the chief of the armed forces and the defense secretary, but even he is not powerful enough to unlock the weapon alone, and the senders (i.e., the Prime Minister and the President) are not powerful enough to issue an individual order that allows the receivers to unlock the weapon. This type of joint responsibility, is essential and routinely exercised in a democracy. However, earlier proposed hierarchical schemes of quantum communication did not contain this particular feature. We can discuss a large number of similar practical situations where HJRSP or a variant of it is essential, but it's not our purpose to provide a long list of practical situation. Rather, we are interested in analyzing the robustness and efficiency of the deterministic HJRSP scheme proposed here.

4 Effect of a set of noise models on the HJRSP scheme

In the recent past, several schemes for classical and quantum communication tasks have been theoretically proposed in the ideal situations, i.e., without considering the effects of noise present in the communication channel. However, it's well understood that in any practical situation, noise would play a crucial role, and the success of a scheme would depend on the noise present in the channel. This fact motivates us to investigate the effect of different

quantum noise models on the HJRSP schemes proposed here. Specifically, in this section we will investigate the effect of AD noise, PD noise, collective dephasing (CD) noise, collective rotation (CR) noise and Pauli noise.

In this section, we aim to analyze the feasibility of the implementation of the proposed deterministic HJRSP scheme in a noisy environment. To quantitatively investigate the effect of noise on a scheme of quantum communication, fidelity, which is a distance based measure, is usually used. Specifically, the fidelity of the state obtained after considering the effect of noise with the reconstructed state in the ideal case can be obtained as

$$F = \langle T | \rho_k | T \rangle. \quad (24)$$

Here, $|T\rangle$ is the quantum state reconstructed after implementing the HJRSP scheme in an ideal situation, while ρ_k is the density matrix of the quantum state obtained after considering the effect of an interaction with the surrounding (i.e., when noise is present). To be precise, the definition of fidelity used here (also used in Refs. [22, 34, 35, 36, 37]) is slightly different from the conventional one, $F'(\sigma, \rho) = \text{Tr} \sqrt{\sigma^{\frac{1}{2}} \rho \sigma^{\frac{1}{2}}}$. Further, in what follows, we will use the strategy adopted in Refs. [22, 34, 35, 36, 37] to study the effect of noise. It is a reasonable assumption that the qubits not traveling through the channel are hardly affected by the noisy environment. Hence, we have not considered the effect of noise on the home qubit, which Alice₁ has prepared and kept for herself. Consider the initial quantum state $\rho = |C\rangle_{S_1 S_2 R_1 R_2 R_3} \langle C|$, where $|C\rangle_{S_1 S_2 R_1 R_2 R_3}$ is the cluster state given in Eq. (1). The transformed density matrix under the effect of AD or PD noisy channel can be expressed as

$$\rho_k = \sum_{i,j,k,l} I_{2,S_1} \otimes E_{i,S_2}^k \otimes E_{j,R_1}^k \otimes E_{k,R_2}^k \otimes E_{l,R_3}^k \rho (I_{2,S_1} \otimes E_{i,S_2}^k \otimes E_{j,R_1}^k \otimes E_{k,R_2}^k \otimes E_{l,R_3}^k)^\dagger, \quad (25)$$

where I_2 is a 2×2 identity matrix, and its application on qubit S_1 corresponds to unaffected home qubit of Alice₁. For the remaining qubits E_J^k are the Kraus operators for AD or PD noise channels with $k \in \{AD, PD\}$ for AD and PD noise, respectively. Here, $J \in \{0, 1\}$ for AD and $J \in \{0, 1, 2\}$ for PD noise models. The Kraus operators for AD and PD noise channels will be described in detail in the following subsections. In the subscripts of the Kraus operators the qubit on which it operates are also mentioned. The same strategy may be used for the investigation of the effect of Pauli channels, where the four Pauli gates (including identity operator) are used to study the errors introduced due to noisy channel. Further, the evolved quantum state under the collective noise models can be described as

$$\rho_k = (I_{2,S_1} \otimes U_k^{\otimes 4}) \rho (I_{2,S_1} \otimes U_k^{\otimes 4})^\dagger, \quad (26)$$

where the subscript k is CD or CR for CD or CR noise channels, and U_k is a 2×2 unitary matrix for either CD or CR noise channels.

In the following subsections, we will analyze the deterministic HJRSP scheme subjected to various noise models after briefly introducing them. Further, we will discuss the dependence of the obtained fidelity expressions on the noise parameters, which quantitatively illustrates the effect of noise on the scheme. Here, we will refrain from considering the effect of noise on the probabilistic HJRSP scheme, which will be discussed elsewhere.

4.1 Amplitude damping (AD) noise:

The AD noise model is represented by the following Kraus operators [3, 34, 38]

$$E_0^{AD} = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\eta_A} \end{bmatrix}, \quad E_1^{AD} = \begin{bmatrix} 0 & \sqrt{\eta_A} \\ 0 & 0 \end{bmatrix}, \quad (27)$$

where η_A ($0 \leq \eta_A \leq 1$) is the decoherence rate and is the probability of energy loss when a travel qubit passes through an AD channel. Specifically, an AD channel simulates the interaction of a quantum system with a vacuum bath. Using Eqs. (24), (25) and (27), we obtain the average fidelity of the quantum state reconstructed by the receivers Bob₂ and Bob₃ as

$$F_{AD}^{\text{Bob}_2} = \frac{(\eta_A^2 - \eta_A + 2) \sin^4(\theta) + (\eta_A^2 - \eta_A + 2) \cos^4(\theta) - 2 \sin^2(\theta) \cos^2(\theta) (\eta_A^3 + (\eta_A - 1) \eta_A^2 \cos(2\phi) - 2\eta_A^2 + \eta_A - 2)}{2(\eta_A^2 + 1)}, \quad (28)$$

and

$$F_{AD}^{\text{Bob}_3} = \frac{\sin^2(\theta) \cos^2(\theta) (-\sqrt{1-\eta_A} \eta_A^2 + (1-\eta_A)^{3/2} \eta_A \cos(2\phi) - \sqrt{1-\eta_A} \eta_A + 2\eta_A + 2\sqrt{1-\eta_A}) + \sin^4(\theta) + \cos^4(\theta)}{\eta_A + 1}, \quad (29)$$

respectively. It would be relevant to mention that the fidelity expressions of the quantum state obtained by other lower power agent Bob₁ are exactly the same as that for Bob₃ in all the noise models discussed in this paper. Hence, the state reconstructed by both the lower power agents will be equally affected by the noise. Henceforward, we will only report the fidelity expressions for Bob₃ and the conclusions deduced from them will be automatically implementable for Bob₁ as well. Further, it would be appropriate to note that the fidelity expressions reported in Eqs. (28) and (29) are averages of fidelity obtained for all the possible cases, i.e., different measurement outcomes of both the senders and the receivers whose cooperation is essential for the reconstruction of the quantum state at the receiver's end. In particular, when the higher (lower) power agent reconstructs a quantum state, we need to compute fidelity expressions for 8 (16) possible combinations of measurement outcomes of all the senders and the other collaborating receivers. In all the following expressions of fidelity and the figures shown here we are explicitly calculating this average fidelity.

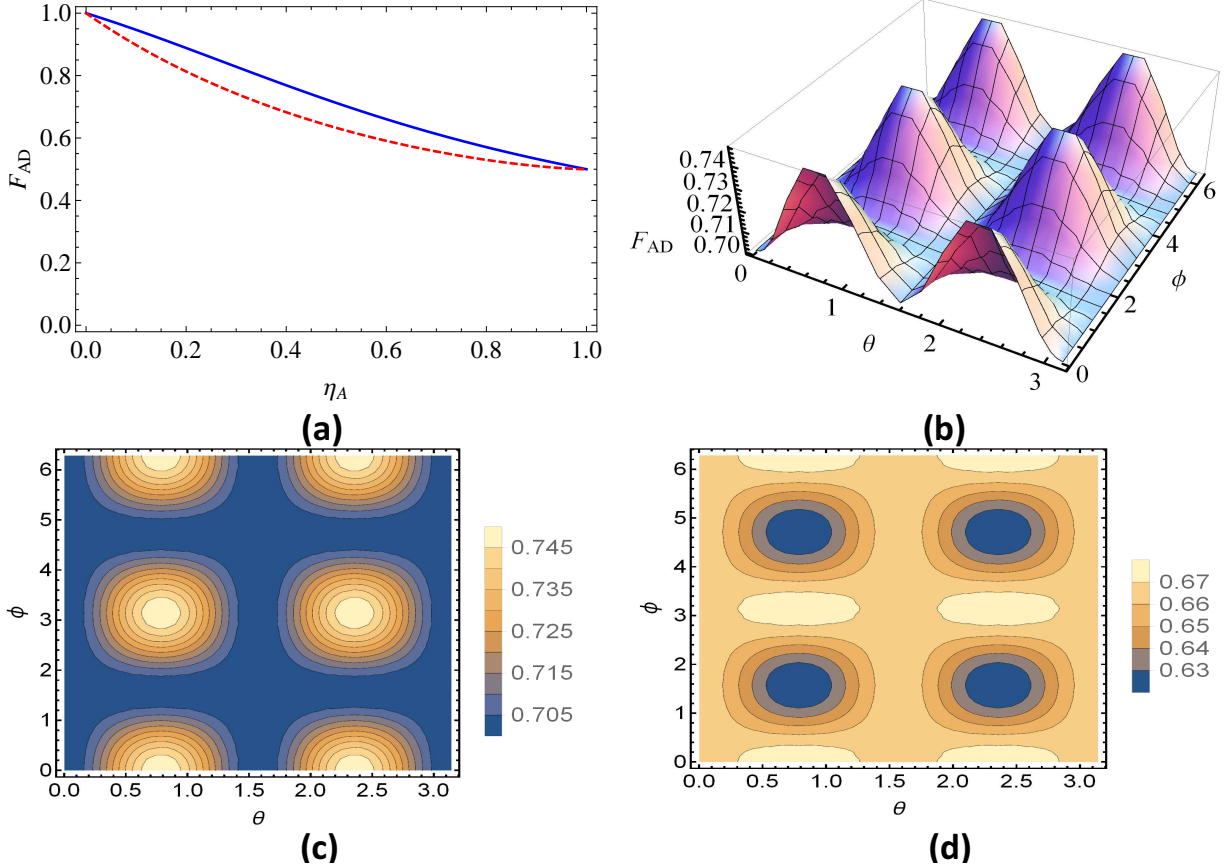


Figure 2: (Color online) In (a), the fidelity F_{AD} is plotted with the decoherence rate η_A considering the state parameters $\theta = \frac{\pi}{4}$ and $\phi = \frac{\pi}{3}$. The smooth (blue) and dotted (red) lines correspond to the fidelities of the quantum states for the higher power agent Bob₂ and the lower power agent Bob₃, respectively. (b) shows three dimensional variation of the fidelity $F_{AD}^{\text{Bob}_2}$ with quantum state parameters, i.e., amplitude (θ) and phase (ϕ) for decoherence rate $\eta_A = \frac{1}{2}$. (c) and (d) show the variation of the average fidelity of the quantum state obtained by the receivers Bob₂ and Bob₃, respectively, with the state parameters considering $\eta_A = \frac{1}{2}$.

The average fidelity (F_{AD}) depends on various parameters, such as amplitude (θ) and phase (ϕ) of the quantum state to be remotely prepared and the decoherence rate (η_A). This dependence is illustrated in Fig. 2. Specifically, Fig. 2 a establishes that the state reconstructed by the higher power agent Bob₂ gets less affected due to noise in comparison to the state reconstructed by the lower power agents. Fig. 2 b illustrates variation of the fidelity with state parameters for the higher power agent considering the decoherence rate $\eta_A = \frac{1}{2}$. It can be inferred from the plot that the obtained fidelity may be higher for certain choices of quantum state to be remotely prepared. These facts can also be illustrated using contour plots. Hereafter, we would only use contour plots to investigate the effect of noise on the deterministic HJRSP scheme. However, here we have shown a corresponding contour plot as well in Fig. 2 c. A slightly different nature of dependence for the average fidelity of the lower power agent on state parameters can be observed in Fig. 2 d.

The case reported here for a dissipative interaction with a vacuum bath, simulated as AD channel, can be extended to generalized amplitude damping [39, 40] and squeezed generalized amplitude damping [39, 40] noise models, where finite temperature reservoir are considered with zero and non-zero squeezing, respectively. The same will be investigated separately.

4.2 Phase damping (PD) noise:

The PD noise model is represented by the following Kraus operators [3, 34, 38]

$$E_0^{PD} = \sqrt{1 - \eta_P} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad E_1^{PD} = \sqrt{\eta_P} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_2^{PD} = \sqrt{\eta_P} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad (30)$$

where η_P ($0 \leq \eta_P \leq 1$) is the decoherence rate. The PD noise simulates an interaction with the surroundings when energy loss is not involved. In presence of this noise, the average fidelities of the quantum state reconstructed by the higher and lower power agents Bob₂ and Bob₃ can be obtained using Eqs. (24), (25) and (30) as

$$F_{PD}^{\text{Bob}_2} = \frac{1}{4} (\eta_P^2 - (\eta_P - 2)\eta_P \cos(4\theta) - 2\eta_P + 4),$$

and

$$F_{PD}^{\text{Bob}_3} = \frac{1}{4} (-\eta_P^3 + (\eta_P^2 - 3\eta_P + 3)\eta_P \cos(4\theta) + 3\eta_P^2 - 3\eta_P + 4), \quad (31)$$

respectively.

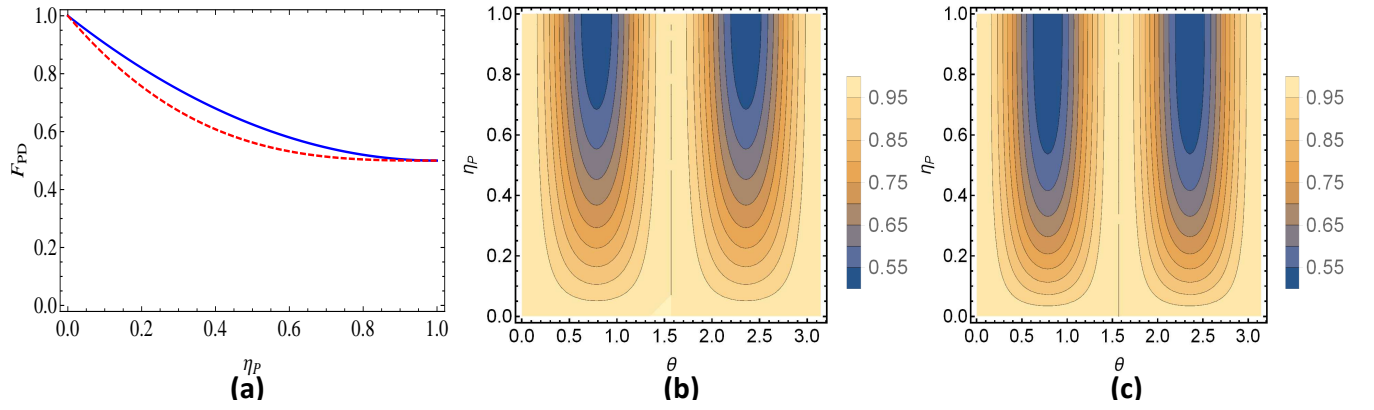


Figure 3: (Color online) In (a), the average fidelity F_{PD} of the quantum state reconstructed by both lower and higher power agents are plotted together with the decoherence rate η_P in the presence of PD noise for $\theta = \frac{\pi}{4}$. The smooth (blue) and dotted (red) lines correspond to the fidelity for the state reconstructed by higher power agent and the lower power agent, respectively. (b) and (c) show the variation of average fidelity with state parameter θ and decoherence rate η_P in contour plots for receivers Bob₂ and Bob₃, respectively. It can be observed that the contour plot for fidelity variation of Bob₂'s quantum state is symmetric with that of Bob₃. However, the higher power agent obtains the state with slightly more fidelity than that of the lower power agent.

It can be observed from the expressions of average fidelity in Eq. (31) that it is phase (ϕ) independent. Consequently, a family of states with the same value of θ undergo the same decoherence, i.e., for a particular value of θ all the states will be affected in the same way for any value of ϕ . Here, in the HJRSP scheme, this corresponds to the fact that any choice of Alice₂ will not affect the state reconstructed by the receivers. Further, we have illustrated the dependence of average fidelity on various parameters in Fig. 3. Specifically, Fig. 3 a shows a similar nature of what was observed in the presence of AD noise in Fig. 2 a, i.e., if the higher power agent Bob₂ reconstructs the state, then the reconstructed state gets less affected by noise in comparison to the cases where the lower power agents reconstruct the state. In Fig. 3 b and c, the contour plots of variation of the average fidelity with the amplitude (θ) and decoherence rate (η_P) are shown for Bob₂ and Bob₃, respectively. The contour plots show that although a symmetric variation in the fidelity is observed, the state reconstructed by the higher power agent is found to be less affected by the noise. Further, it is observed that for certain values of different parameters we may obtain a unit fidelity even in the noisy environment.

4.3 Collective dephasing (CD) noise:

The CD noise model is represented by the following unitary (phase) operator

$$U_{CD} = \begin{bmatrix} 1 & 0 \\ 0 & \exp(i\Phi) \end{bmatrix}, \quad (32)$$

where Φ is the noise parameter that may change with time, but remains the same at an instant for all the qubits traveling simultaneously through the noisy channel. It should be noted here that the collective noise models consider a coherent effect of interaction on all the travel qubits. The average fidelities for the state reconstructed by Bob₂ and Bob₃ can be obtained under CD noise model as

$$\begin{aligned} F_{CD}^{\text{Bob}_2} &= \frac{1}{16}(-\cos(4\theta - 2\Phi) - \cos(4\theta + 2\Phi) + 2\cos(4\theta) + 2\cos(2\Phi) + 14), \\ F_{CD}^{\text{Bob}_3} &= \sin^2(\theta) \cos^2(\theta)(\cos(\Phi) + \cos(3\Phi)) + \sin^4(\theta) + \cos^4(\theta), \end{aligned} \quad (33)$$

respectively.

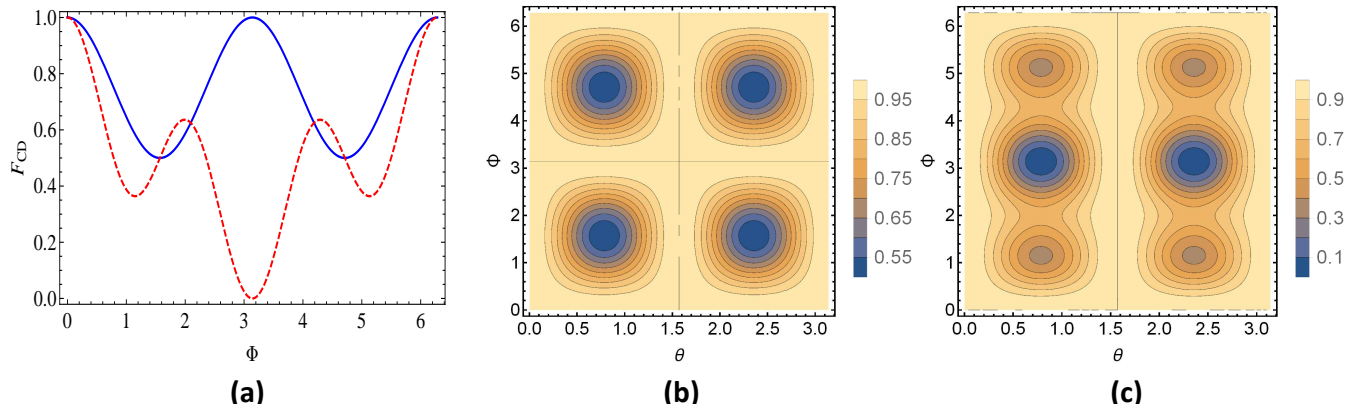


Figure 4: (Color online) In (a), the average fidelity F_{CD} is shown to vary with the noise parameter (Φ) of CD noise for $\theta = \frac{\pi}{4}$. The smooth (blue) and dotted (red) lines correspond to the fidelities of the quantum states obtained by the higher and lower power agents, respectively. (b) and (c) show the variation of the average fidelity with a noise parameter and the amplitude of the state to be remotely prepared for the receivers Bob₂ and Bob₃, respectively.

From the above expressions, it can be observed that the average fidelity is free from the phase of the quantum state to be remotely prepared. Hence, the fidelity of the obtained state will only depend on the noise parameter and the amplitude of the state to be reconstructed. To elaborate the dependence explicitly, we have plotted the average fidelity (F_{CD}) as a function of amplitude (θ) and the noise parameter (Φ) in Fig. 4. Specifically, in Fig. 4 a, the average fidelity is shown to vary with the noise parameter for a family of states with an arbitrary value of phase angle and amplitude $\theta = \frac{\pi}{4}$. Here, it is interesting to note that, unlike the previous cases when the HJRSP scheme was subjected to AD and PD noise models, even the lower power agent can acquire the quantum state with higher fidelity than that of the higher power agent. Specifically, in Fig. 4 a, in the vicinity of $\Phi = \frac{\pi}{2}$ the state reconstructed by a lower power agent is found to be less affected by noise in comparison to the same state reconstructed by a higher power agent. Further, when the noise parameter $\Phi = \pi$ then Bob₂ recovers unaffected state, i.e., the state with unit fidelity, while Bob₃ can reconstruct the quantum state with negligible fidelity. Fig. 4 b and c further illustrate the same facts through contour plots and manifests the effect of noise on all the possible quantum states that can be remotely prepared. It can be observed from the plots that unit fidelity can be obtained in some cases. Except a few values of the parameters, in general, the higher power agent can extract higher fidelity quantum state.

4.4 Collective rotation (CR) noise:

The CR noise model is represented by a unitary rotation operator

$$U_{CR} = \begin{bmatrix} \cos \Theta & -\sin \Theta \\ \sin \Theta & \cos \Theta \end{bmatrix}, \quad (34)$$

where Θ is the noise parameter and have the same effect and property as Φ in the CD noise model. The expressions for the average fidelities of the quantum state reconstructed by Bob₂ and Bob₃ are a bit complex and to ensure that the flow of the paper is not disturbed, they are reported in Appendix A in Eq. (39).

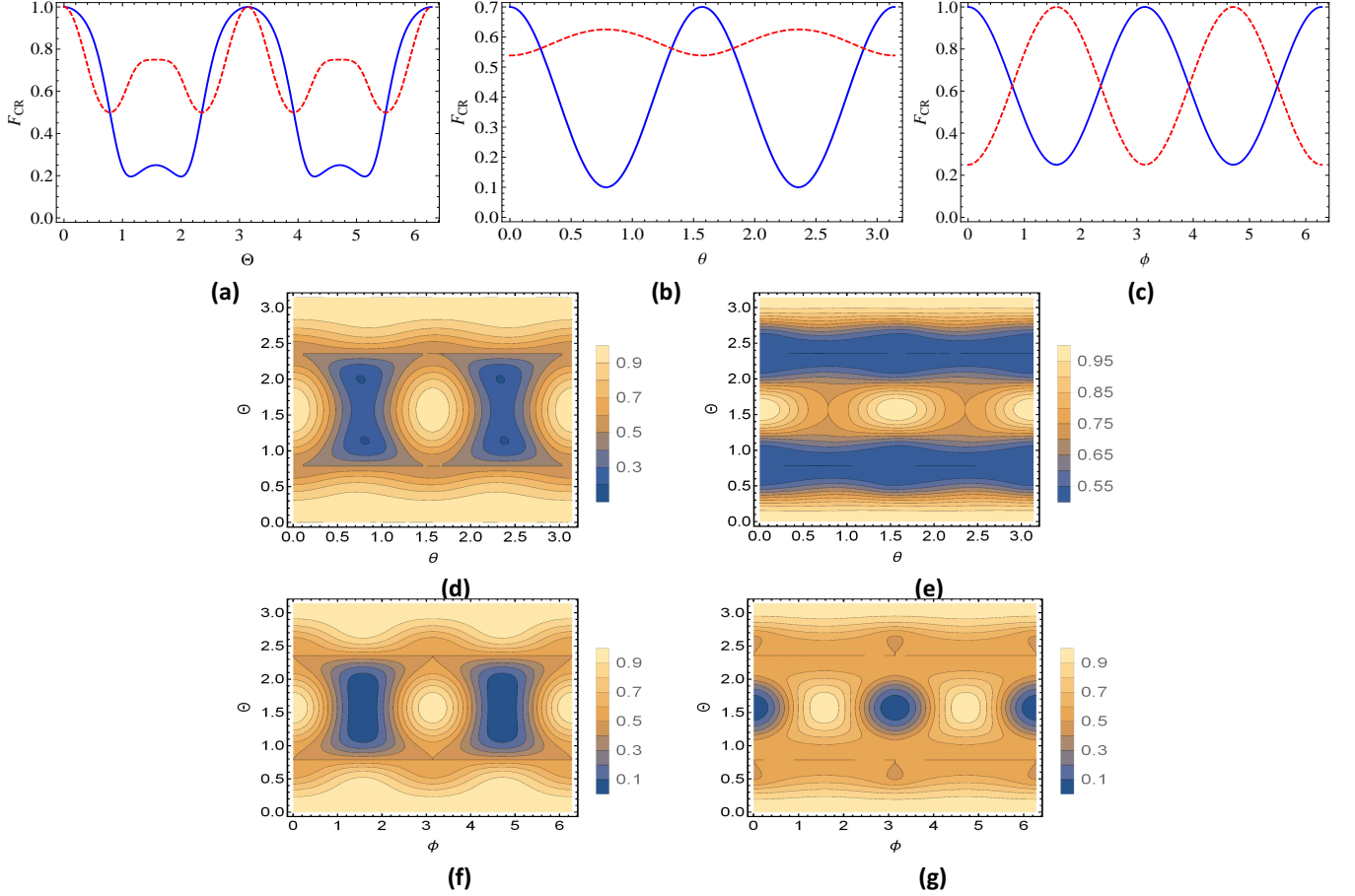


Figure 5: (Color online) In (a), the average fidelity F_{CR} is plotted together for both higher and lower power agents with the noise parameter Θ for state parameters $\theta = \pi/4$ and $\phi = \pi/3$. (b) illustrates the dependence of the average fidelity F_{CR} of the reconstructed state for the agents on the amplitude parameter θ for $\phi = \pi/2$ and noise parameter $\Theta = \pi/3$. In (c), the average fidelity is plotted with the phase angle ϕ for $\theta = \pi/3$ and $\Theta = \pi/2$. In (a)-(c) the smooth (blue) and dotted (red) lines correspond to the fidelities of state reconstructed by higher power agent Bob₂ and the lower power agent Bob₃, respectively. (d) and (e) show the variation of average fidelity with noise parameter and amplitude parameter considering $\phi = \pi/3$ for the receivers Bob₂ and Bob₃, respectively. Similarly, (f) and (g) demonstrate the variation for $\theta = \pi/4$ with noise parameter and phase angle for the higher and lower power agents, respectively.

To express the dependence of average fidelity of the state received by both higher and lower power agents on various parameters we have plotted the fidelity expressions in Fig. 5. Specifically, Fig. 5 a shows the variation in the average fidelity F_{CR} with the noise parameter (Θ) for a particular quantum state with $\theta = \pi/4$ and $\phi = \pi/3$ to be remotely prepared. Interestingly, for a few specific values of the noise parameter (e.g., for $\frac{\pi}{3} < \Theta < \frac{2\pi}{3}$), the higher power agent can reconstruct the state with lesser fidelity compared to the state obtained by the lower power agent. In contrast to the CD noise case, here lower power agent can also reconstruct a state with unit fidelity. For a particular value of noise parameter, i.e., $\Theta = \pi/3$, the class of states with phase angle $\frac{\pi}{2}$ show that they are not equally affected due to noise (cf. Fig. 5 b). In fact, if the state is reconstructed by the higher power agent then the value of the amplitude parameter matters more than the cases when lower power agents choose to do so. Further, it also shows that for $\theta = \pi/2$, higher power agent Bob₂ achieves the maximum fidelity of the received state, whereas the fidelity of the state obtained by the lower power agent Bob₃ could never reach that limit. Fig. 5 c shows the fidelity variation with phase angle ϕ , and a complimentary nature for the fidelity of the quantum state obtained by higher and lower power agents is observed, i.e., it is found that if the fidelity for one increases,

it decreases for the other. The contour plots in Fig. 5 d-g show that some specific choice of initial quantum states can be remotely prepared with unit fidelity or fidelity close to 1 in the CR noisy environment for particular values of the noise parameter.

The nature observed here for the collective noise model, i.e., the effect of collective noise is in contrast of that in the AD and PD channels, is consistent with a few earlier observations [35, 41]. Specifically, in Ref. [35], it has been observed that single qubits perform better as decoy qubits in secure quantum communication than entangled qubits when subjected to AD and PD noise; whereas completely opposite results have been obtained in collective noise, i.e., a few Bell states were shown to be decoherence free. Similarly, single-qubit-based secure quantum communication schemes have been found robust in AD and PD channels while their entangled-state-based counterparts were found to perform better in the presence of collective noise [41].

4.5 Pauli (P) noise:

There are four well known Pauli matrices that are frequently used in quantum information. Just to introduce the notation followed in this section, we note that the Pauli matrices are described in the following manner:

$$\sigma_1 \equiv \sigma_x \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 \equiv \sigma_y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 \equiv \sigma_z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \sigma_4 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \quad (35)$$

These operators are relevant for the present study as a quantum state described by the density operator ρ , evolves under Pauli noise model (in Pauli channel) as described in Eq. (25). Specifically, in Pauli channels, a single qubit state will evolve as

$$\rho' = \sum_{i=1}^4 E_i^P \rho E_i^{P\dagger}, \quad (36)$$

where $E_i^P = \sqrt{p_i} \sigma_i$: p_i is the probability of i th type of error, which is modeled by Pauli operator (error operator) σ_i defined in Eq. (35), and $\sum_i^4 p_i = 1$. To visualize that p_i is the probability of the error modeled by σ_i , we may expand Eq. (36) as

$$\rho' = p_1 \sigma_1 \rho \sigma_1 + p_2 \sigma_2 \rho \sigma_2 + p_3 \sigma_3 \rho \sigma_3 + (1 - p_1 - p_2 - p_3) \rho, \quad (37)$$

which clearly shows that the density operator ρ undergoes a bit-flip (modeled by σ_1) with the probability p_1 , a combined bit and phase-flip (modeled by σ_2) with probability p_2 and a phase flip (modeled by σ_3) with the probability p_3 . However, ρ remains unchanged with the probability $(1 - p_1 - p_2 - p_3) = p_4$, which implies an error free channel. The Pauli channel is widely discussed in the literature [42, 43, 44], and its name originates from the error operators of the channel. Specifically, the most general Pauli channel described by (37) reduces to (i) a bit flip channel when $p_2 = p_3 = 0$, (ii) a bit-phase flip channel when $p_1 = p_3 = 0$, (iii) a phase flip channel when $p_1 = p_2 = 0$, and (iv) a noise free channel or identity channel when $p_1 = p_2 = p_3 = 0$. Similarly, we may define a depolarizing noise channel when all the bit-flip, phase-flip, and bit-phase-flip errors occur with equal probability $p_1 = p_2 = p_3 = p \leq \frac{1}{3}$, and the state remains unchanged with the remaining probability.

Above discussion shows that if we can obtain fidelity for the most general Pauli channel described by (37), we can obtain the fidelities for evolution under specific noise models as special cases of that. Keeping this in mind, we have obtained following analytic expressions for the average fidelity of the quantum state remotely prepared by the proposed HJRSP scheme when the quantum state to be prepared remotely is evolved under most general Pauli noise model, and either Bob₂ or Bob₃ reconstructs the state:

$$\begin{aligned} F_P^{\text{Bob}_2} &= \frac{1}{(p_1+p_2)^2+(p_3+p_4)^2} ((p_1^2+p_2^2+p_1(2p_2-p_3-p_4)-p_2(p_3+p_4)+(p_3+p_4)^2) \cos^4(\theta) \\ &+ 2 \cos^2(\theta)(p_3^4-p_1^3(p_3-3p_4)+p_2^3(3p_3-p_4)+p_4^4+p_2(p_3+p_4)^3+3p_2^2(p_3^2+p_4^2) \\ &+ p_1(p_2^2p_4+(p_3+p_4)^3)+p_1^2(p_2p_3+3(p_3^2+p_4^2))+(p_1^4+p_1^2(-2p_2^2+(p_3-p_4)^2) \\ &- 2p_1(p_3-p_4)(p_2(-p_3+p_4)+(p_3+p_4)^2)+p_2(p_3^3+p_2(p_3-p_4)^2+2(p_3-p_4) \\ &\times (p_3+p_4)^2)) \cos(2\phi) \sin^2(\theta) + (p_1^2+p_2^2+p_1(2p_2-p_3-p_4)-p_2(p_3+p_4) \\ &+ (p_3+p_4)^2) \sin^4(\theta) + \frac{1}{2}(p_1^2(5p_2+6p_3)p_4-2p_3p_4(-3p_2^2+p_3p_4)+p_1p_2(5p_2p_3 \\ &+ 2(p_3+p_4)^2)) \sin^2(2\theta), \\ F_P^{\text{Bob}_3} &= (((p_1+p_2)^2+(p_3+p_4)^2) \cos^4(\theta) + 2 \cos^2(\theta)(p_3(p_1^3-(p_2-p_3)^2(p_2+p_3) \\ &+ p_1(11p_2^2+3p_3^2))+p_2(p_1^2+3p_2^2)p_4+3p_2^2p_4^2+(3p_1+p_2)p_4^3+p_4^4+(p_1-p_2 \\ &- p_3+p_4)(p_1^3-p_1^2p_2+p_2(p_2^2+(p_3-p_4)(3p_3+p_4))-p_1(p_2^2+(p_3-p_4)(p_3+3p_4))) \\ &\times \cos(2\phi) \sin^2(\theta) + ((p_1+p_2)^2+(p_3+p_4)^2) \sin^4(\theta) + \frac{1}{2}(5p_1^3p_4+p_1^2(5p_3(p_2+p_3) \\ &+ 4p_3p_4+7p_4^2)+p_3p_4(12p_2^2+7p_2(p_3+p_4)+2(p_3-p_4)(p_3+p_4))+p_1(7p_2^2p_4 \\ &+ 5p_3p_4(p_3+p_4)+2p_2(p_3+p_4)(5p_3+3p_4))) \sin^2(2\theta)). \end{aligned} \quad (38)$$

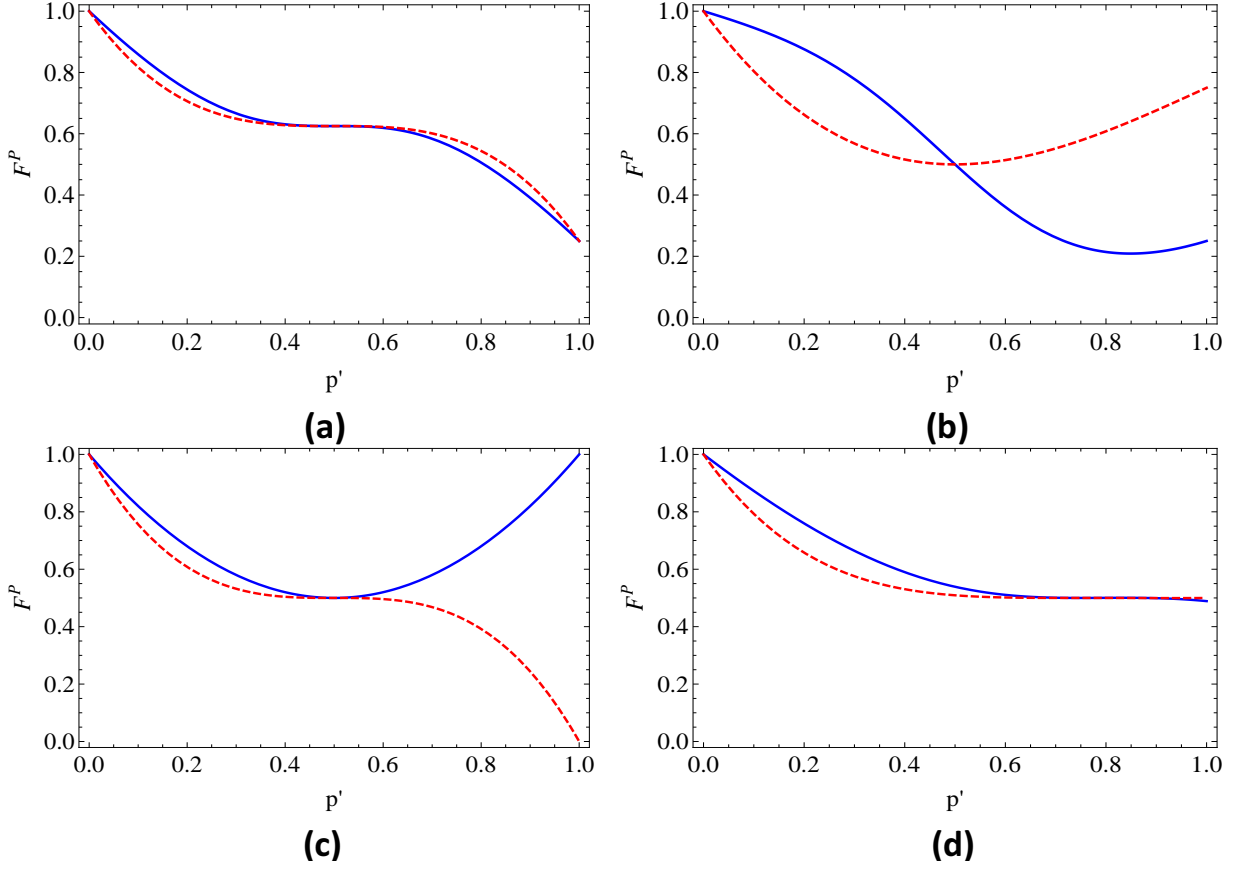


Figure 6: (Color online) The average fidelity of the quantum state reconstructed by both the higher and lower power agents when subjected to Pauli noise are shown. (a), (b), (c), and (d) correspond to bit-flip, bit-phase-flip, phase-flip and depolarizing channels, respectively. In all these plots, $\theta = \pi/4$ and $\phi = \pi/3$. The smooth (blue) and dotted (red) lines correspond to the fidelities of the quantum states obtained by the higher power agent and the lower power agent, respectively.

To illustrate the effect of Pauli noise, we have plotted the average fidelity (F_P) under the Pauli noise model for the state parameters $\theta = \pi/4$ and $\phi = \pi/3$ in Fig. 6. Here, we have used a depolarizing channel, where the total probability of error for bit, phase and bit-phase flip errors is p' , and consequently, $\frac{p'}{3}$ is the probability for each kind of error. Specifically, in Fig. 6 (a), the average fidelity F_P is shown in case of bit-flip errors. It can be observed that the fidelities of the state obtained by Bob₂ and Bob₃ vary almost similarly and decay to the lowest value when the probability of errors p' is unity. Similarly, in Fig. 6 (b), the average fidelity of the state reconstructed by both higher and lower power agents in bit-phase-flip channel is shown. Interestingly, it has been observed that for higher probability of errors, i.e., $p' > 0.5$, even higher power agent Bob₂ has less fidelity of the reconstructed state than that of the lower power agent. In fact, beyond this value of probability of errors a revival in the fidelity for the lower power agent can be seen. In Fig. 6 (c), variation of the average fidelity F_P with the probability of phase-flip error is illustrated. In contrast with Fig. 6 (b), here the fidelity obtained by the higher power agent is revived and reached to unity for maximum probability of error. Further, the fidelity of the state obtained by the higher power agent is always higher than that of the lower power agent, which shows a gradual decay with increasing probability of error (cf. red (dashed) line in Fig. 6 (c)). Finally, in Fig. 6 (d), we have plotted the average fidelity F_P under the effect of depolarizing channel, where the fidelity of states obtained by both the receivers converges for $0.6 < p' < 0.9$, and after that it decays for the higher power agent.

5 Conclusions

In the present paper we have introduced the notion of HJRSP and have proposed a protocol for deterministic HJRSP using 5-qubit cluster state of the form (1). The scheme is also illustrated through Fig. 1. It is unique in nature because it allows joint preparability (more than one sender), a feature which is desirable in practical situations

as illustrated in Sec. 3, but not present in any of the existing schemes of hierarchical quantum communication. Further, in the present work, the effect of various noise models on the proposed HJRSP scheme has been investigated in detail. In contrast, in none of the existing proposals for hierarchical quantum communication, the effect of noise has been discussed. However, in a practical situation, it is impossible to circumvent the existence of noise. Thus, the present study not only makes it more realistic, it also yields an extremely relevant and essential feature of hierarchical quantum communication. Further, in addition to the deterministic HJRSP, a protocol for probabilistic HJRSP using a non-maximally entangled cluster state is also introduced here. Usually, a reduction of required quantum resources is expected in probabilistic RSP. However, it was not obtained here due to use of a non-maximally entangled state.

The study of the effect of different noise models has led to many interesting results. Specifically, it is observed that the quantum state that the higher power agent and the lower power agent can reconstruct may get decohered due to interaction with its surroundings, which have been analyzed and discussed in detail in the previous section. In brief, the higher power agent can reconstruct the quantum state less affected due to AD and PD noise than that of the lower power agent. However, when the travel qubits are subjected to collective or Pauli noise, lower power agent may also get better performance than that of the higher power agent. Interestingly, a few specific quantum states can be remotely prepared in an unaffected manner, even in the presence of noise. Here, we have restricted ourselves to Markovian noise. The effect of non-Markovian type of noise will be reported elsewhere.

It would be interesting to study controlled versions of the hierarchical quantum communication protocols proposed here and in the recent past, as a variety of applications of the various controlled quantum communication protocols have been widely discussed [36]. We expect that due to the wide applicability of the hierarchical schemes the proposed scheme may be of interest to experimentalists and in the near future the results reported here will be experimentally verified.

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Appendix A

The average fidelity expressions of the quantum states reconstructed by the higher and lower power agents when the HJRSP scheme is subjected to CR noise are obtained as

$$\begin{aligned}
F_{CR}^{\text{Bob}_2} &= \frac{1}{8((3+\cos(4\Theta))^2 - 4\cos^2(\Theta)(\cos(4\Theta-2\theta)+3\cos(2\theta))^2 \cos^2(\phi) \sin^2(\Theta))} \{ (3 + \cos^2(4\Theta))(5 + 3\cos(4\Theta)) \cos^4(\Theta) \\
&- (42 + 88\cos(2\Theta) + 36\cos(4\Theta) + 23\cos(6\Theta) + 2\cos(8\Theta) + \cos(10\Theta) + 9\cos(6\Theta - 4\theta) \\
&+ 4\cos(8\Theta - 4\theta) + 2\cos(10\Theta - 4\theta) + 7\cos(2(\Theta - 2\theta)) + 18\cos(4(\Theta - \theta)) + 24\cos(4\theta) \\
&+ 2\cos(4(\Theta + \theta)) - \cos(2(\Theta + 2\theta)) - \cos(6\Theta + 4\theta)) \cos(2\phi) \sin^2(\Theta) - 16(5\cos(2\Theta) + \\
&+ \cos(6\Theta)) \cos^3(\theta) \sin^3(2\Theta) \sin(\theta) + 2\cos^2(\Theta)(78 + 14\cos(2\Theta) + 40\cos(4\Theta) - 9\cos(6\Theta) \\
&+ 10\cos(8\Theta) - 5\cos(10\Theta)) \cos^2(\theta) \sin^2(\theta) + 16(5\cos(2\Theta) + \cos(6\Theta)) \cos(\theta) \sin^3(2\Theta) \sin^3(\theta) \\
&+ (3 + \cos^2(4\Theta))(5 + 3\cos(4\Theta)) \sin^4(\theta) \} ,
\end{aligned}$$

and

$$F_{CR}^{\text{Bob}_3} = [A + B + C + D + E + F + G + H] , \quad (39)$$

where

$$\begin{aligned}
A &= \{1/[(8 - \cos(2\Theta - \phi) + \cos(6\Theta - \phi) + \cos(4\Theta - 2\theta - \phi) - \cos(4\Theta + 2\theta - \phi) - \cos(2\Theta + \phi) \\
&+ \cos(6\Theta + \phi) + \cos(4\Theta - 2\theta + \phi) - \cos(4\Theta + 2\theta + \phi) + 2\sin(4\Theta - 2\theta) + 2\sin(4\Theta + 2\theta) \\
&- 2\sin(2\Theta - 2\theta - \phi) - 2\sin(2\Theta + 2\theta - \phi) - 2\sin(2\Theta - 2\theta + \phi) - 2\sin(2\Theta + 2\theta + \phi)]\} \\
&\times \{\cos^8(\Theta) + \cos^4(\theta)\sin^8(\Theta) - 2\cos^2(\theta)\cos(2\phi)\sin^8(\Theta)\sin^2(\theta) + \sin^8(\Theta)\sin^4(\theta) \\
&+ 2\cos^4(\Theta)\sin^4(\Theta)(\cos^4(\theta) + 2\cos^3(\theta)(1 - 4\cos(\phi) + \cos(2\phi))\sin(\theta) + \cos^2(\theta)\cos(2\phi)\sin^2(\theta) \\
&- 4\cos(\theta)\cos^2(\phi)\sin^3(\theta) + \sin^4(\theta)) + 2\cos^2(\Theta)\sin^6(\Theta)(2\cos^4(\theta)\cos(\phi) - 4\cos^3(\theta)\cos^2(\phi)\sin(\theta) \\
&+ \cos^2(\theta)(3 + \cos(2\phi))\sin^2(\theta) - 4\cos(\theta)\cos(\phi)\sin^3(\theta) + 4\cos^2(\frac{\phi}{2})\sin^4(\theta)) + \cos(\Theta)\cos(2\theta)\sin^7(\Theta) \\
&\times (-2 + 2\cos(\phi)(-1 + \sin(2\theta))) + \frac{3}{32}\sin^4(2\Theta)\sin^2(2\theta) + 2\cos^7(\Theta)\sin(\Theta)(-\cos^2(\theta)(-1 + \cos(\phi)) \\
&+ (-1 + \cos(\phi))\sin^2(\theta) + \cos(\phi)\sin(2\theta)) - 4\cos^6(\Theta)\sin^2(\Theta)(\cos^4(\theta)(-1 + \cos(\phi)) + \cos(\phi)\sin^4(\theta) \\
&+ 4\cos(\theta)\cos(\phi)\sin^3(\theta)\sin^2(\frac{\phi}{2}) - \cos^2(\theta)\sin^2(\theta)\sin^2(\phi)) + \cos^3(\Theta)\sin^5(\Theta)(-2\cos(2\theta)(1 + \cos(\phi)) \\
&+ \cos(\phi)(2 + 2\cos(4\theta) - \cos(4\theta - \phi) + 2\cos(\phi) - \cos(4\theta + \phi) - 2\sin(2\theta) + \sin(4\theta - \phi) + \sin(4\theta + \phi))) \\
&+ \cos^5(\Theta)\sin^3(\Theta)(-2\cos(2\theta)(-1 + \cos(\phi)) + \cos(\phi)(-2 - 2\cos(4\theta) + \cos(4\theta - \phi) - 2\cos(\phi) \\
&+ \cos(4\theta + \phi) - \sin(4\theta) + \sin(4\theta - \phi) + \sin(4\theta + \phi)))\}, \\
B &= \{1/[8 + \cos(2\Theta - \phi) - \cos(6\Theta - \phi) - \cos(4\Theta - 2\theta - \phi) + \cos(4\Theta + 2\theta - \phi) + \cos(2\Theta + \phi) \\
&- \cos(6\Theta + \phi) - \cos(4\Theta - 2\theta + \phi) + \cos(4\Theta + 2\theta + \phi) + 2\sin(4\Theta - 2\theta) + 2\sin(4\Theta + 2\theta) \\
&+ 2\sin(2\Theta - 2\theta - \phi) + 2\sin(2\Theta + 2\theta - \phi) + 2\sin(2\Theta - 2\theta + \phi) + 2\sin(2\Theta + 2\theta + \phi)]\} \\
&\times \{\cos^8(\Theta) + \cos^4(\theta)\sin^8(\Theta) - 2\cos^2(\theta)\cos(2\phi)\sin^8(\Theta)\sin^2(\theta) + \sin^8(\Theta)\sin^4(\theta) \\
&+ 2\cos^7(\Theta)\sin(\Theta)(\cos^2(\theta)(1 + \cos(\phi)) - 2\cos(\theta)\cos(\phi)\sin(\theta) - (1 + \cos(\phi))\sin^2(\theta)) \\
&+ 2\cos^4(\Theta)\sin^4(\Theta)(\cos^4(\theta) + 2\cos^3(\theta)(1 + 4\cos(\phi) + \cos(2\phi))\sin(\theta) + \cos^2(\theta)\cos(2\phi)\sin^2(\theta) \\
&- 4\cos(\theta)\cos^2(\phi)\sin^3(\theta) + \sin^4(\theta)) + 2\cos^2(\Theta)\sin^6(\Theta)(-2\cos^4(\theta)\cos(\phi) - 4\cos^3(\theta)\cos^2(\phi)\sin(\theta) \\
&+ \cos^2(\theta)(3 + \cos(2\phi))\sin^2(\theta) + 4\cos(\theta)\cos(\phi)\sin^3(\theta) - 2(-1 + \cos(\phi))\sin^4(\theta)) - \cos(\Theta)\cos(2\theta)\sin^7(\Theta) \\
&\times (2 + 2\cos(\phi)(-1 + \sin(2\theta))) + \frac{3}{32}\sin^4(2\Theta)\sin^2(2\theta) + 4\cos^6(\Theta)\sin^2(\Theta)(\cos^4(\theta)(1 + \cos(\phi)) \\
&+ 4\cos(\theta)\cos^2(\frac{\phi}{2})\cos(\phi)\sin^3(\theta) + \cos(\phi)\sin^4(\theta) + \cos^2(\theta)\sin^2(\theta)\sin^2(\phi)) + \cos^3(\Theta)\sin^5(\Theta)(2\cos(2\theta) \\
&\times (-1 + \cos(\phi)) + \cos(\phi)(-2 - 2\cos(4\theta) - \cos(4\theta - \phi) + 2\cos(\phi) - \cos(4\theta + \phi) + 2\sin(2\theta) + \sin(4\theta - \phi) \\
&+ \sin(4\theta + \phi))) + \cos^5(\Theta)\sin^3(\Theta)(2\cos(2\theta)(1 + \cos(\phi)) + \cos(\phi)(2 + 2\cos(4\theta) + \cos(4\theta - \phi) - 2\cos(\phi) \\
&+ \cos(4\theta + \phi) + \sin(4\theta) + \sin(4\theta - \phi) + \sin(4\theta + \phi)))\}, \\
C &= \{1/[-8 - \cos(2\Theta - \phi) + \cos(6\Theta - \phi) - \cos(4\Theta - 2\theta - \phi) + \cos(4\Theta + 2\theta - \phi) - \cos(2\Theta + \phi) \\
&- \cos(6\Theta + \phi) + \cos(4\Theta - 2\theta + \phi) + \cos(4\Theta + 2\theta + \phi) + 2\sin(4\Theta - 2\theta) + 2\sin(4\Theta + 2\theta) \\
&+ 2\sin(2\Theta - 2\theta - \phi) + 2\sin(2\Theta + 2\theta - \phi) + 2\sin(2\Theta - 2\theta + \phi) + 2\sin(2\Theta + 2\theta + \phi)]\} \\
&\times \{\cos^8(\Theta) + \cos^4(\theta)\sin^8(\Theta) - 2\cos^2(\theta)\cos(2\phi)\sin^8(\Theta)\sin^2(\theta) + \sin^8(\Theta)\sin^4(\theta) \\
&- 2\cos^7(\Theta)\sin(\Theta)(\cos^2(\theta)(1 + \cos(\phi)) - 2\cos(\theta)\cos(\phi)\sin(\theta) - (1 + \cos(\phi))\sin^2(\theta)) \\
&+ 2\cos^4(\Theta)\sin^4(\Theta)(\cos^4(\theta) + 4\cos^3(\theta)\cos^2(\phi)\sin(\theta) + \cos^2(\theta)\cos(2\phi)\sin^2(\theta) \\
&- 2\cos(\theta)(1 + 4\cos(\phi) + \cos(2\phi))\sin^3(\theta) + \sin^4(\theta)) + 2\cos^2(\Theta)\sin^6(\Theta)(-2\cos^4(\theta)(-1 + \cos(\phi)) \\
&- 4\cos^3(\theta)\cos(\phi)\sin(\theta) + \cos^2(\theta)(3 + \cos(2\phi))\sin^2(\theta) + 4\cos(\theta)\cos^2(\phi)\sin^3(\theta) - 2\cos(\phi)\sin^4(\theta)) \\
&+ \frac{3}{32}\sin^4(2\Theta)\sin^2(2\theta) - 2\cos(\Theta)\cos(2\theta)\sin^7(\Theta)(-1 + \cos(\phi)(1 + \sin(2\theta))) + 4\cos^6(\Theta)\sin^2(\Theta) \\
&\times (\cos^4(\theta)\cos(\phi) - 4\cos^3(\theta)\cos^2(\frac{\phi}{2})\cos(\phi)\sin(\theta) + 2\cos^2(\frac{\phi}{2})\sin^4(\theta) + \cos^2(\theta)\sin^2(\theta)\sin^2(\phi)) \\
&- \cos^3(\Theta)\sin^5(\Theta)(2\cos(2\theta)(-1 + \cos(\phi)) + \cos(\phi)(2 + 2\cos(4\theta) + \cos(4\theta - \phi) - 2\cos(\phi) + \cos(4\theta + \phi) \\
&+ 2\sin(2\theta) - \sin(4\theta - \phi) - \sin(4\theta + \phi) + \cos^5(\Theta)\sin^3(\Theta)(-2\cos(2\theta)(1 + \cos(\phi)) + \cos(\phi)(2 + 2\cos(4\theta) \\
&+ \cos(4\theta - \phi) - 2\cos(\phi) + \cos(4\theta + \phi) + \sin(4\theta) + \sin(4\theta - \phi) + \sin(4\theta + \phi)))\}, \\
D &= \{1/[8 - \cos(2\Theta - \phi) + \cos(6\Theta - \phi) - \cos(4\Theta - 2\theta - \phi) + \cos(4\Theta + 2\theta - \phi) - \cos(2\Theta + \phi) \\
&+ \cos(6\Theta + \phi) - \cos(4\Theta - 2\theta + \phi) + \cos(4\Theta + 2\theta + \phi) - 2\sin(4\Theta - 2\theta) - 2\sin(2\Theta + \theta)] \\
&+ 2\sin(2\Theta - 2\theta - \phi) + 2\sin(2\Theta + 2\theta - \phi) + 2\sin(2\Theta - 2\theta + \phi) + 2\sin(2\Theta + 2\theta + \phi)]\} \\
&\times \{\cos^8(\Theta) + \cos^4(\theta)\sin^8(\Theta) - 2\cos^2(\theta)\cos(2\phi)\sin^8(\Theta)\sin^2(\theta) + \sin^8(\Theta)\sin^4(\theta) \\
&+ 2\cos^7(\Theta)\sin(\Theta)(\cos^2(\theta)(-1 + \cos(\phi)) - 2\cos(\theta)\cos(\phi)\sin(\theta) - (-1 + \cos(\phi))\sin^2(\theta)) \\
&+ 2\cos^4(\Theta)\sin^4(\Theta)(\cos^4(\theta) + 4\cos^3(\theta)\cos^2(\phi)\sin(\theta) + \cos^2(\theta)\cos(2\phi)\sin^2(\theta) \\
&- 2\cos(\theta)(1 - 4\cos(\phi) + \cos(2\phi))\sin^3(\theta) + \sin^4(\theta)) + 2\cos^2(\Theta)\sin^6(\Theta)(2\cos^4(\theta)(1 + \cos(\phi)) \\
&+ 4\cos^3(\theta)\cos(\phi)\sin(\theta) + \cos^2(\theta)(3 + \cos(2\phi))\sin^2(\theta) + 4\cos(\theta)\cos^2(\phi)\sin^3(\theta) + 2\cos(\phi)\sin^4(\theta)) \\
&+ \frac{3}{32}\sin^4(2\Theta)\sin^2(2\theta) + 2\cos(\Theta)\cos(2\theta)\sin^7(\Theta)(1 + \cos(\phi)(1 + \sin(2\theta))) - 4\cos^6(\Theta)\sin^2(\Theta) \\
&\times (\cos^4(\theta)\cos(\phi) + (-1 + \cos(\phi))\sin^4(\theta) - 4\cos^3(\theta)\cos(\phi)\sin(\theta)\sin^2(\frac{\phi}{2}) - \cos^2(\theta)\sin^2(\theta)\sin^2(\phi)) \\
&+ \cos^3(\Theta)\sin^5(\Theta)(2\cos(2\theta)(1 + \cos(\phi)) + \cos(\phi)(2 + 2\cos(4\theta) - \cos(4\theta - \phi) + 2\cos(\phi) - \cos(4\theta + \phi) \\
&+ 2\sin(2\theta) + \sin(4\theta - \phi) + \sin(4\theta + \phi) + \cos^5(\Theta)\sin^3(\Theta)(2\cos(2\theta)(-1 + \cos(\phi)) + \cos(\phi)(-2 - 2\cos(4\theta) \\
&+ \cos(4\theta - \phi) - 2\cos(\phi) + \cos(4\theta + \phi) - \sin(4\theta) + \sin(4\theta - \phi) + \sin(4\theta + \phi)))\},
\end{aligned}$$

$$\begin{aligned}
E &= \{1/[-8 - \cos(2\Theta - \phi) + \cos(6\Theta - \phi) + \cos(4\Theta - 2\theta - \phi) - \cos(4\Theta + 2\theta - \phi) - \cos(2\Theta + \phi) \\
&+ \cos(6\Theta + \phi) + \cos(4\Theta - 2\theta + \phi) - \cos(4\Theta + 2\theta + \phi) + 2\sin(4\Theta - 2\theta) + 2\sin(2(2\Theta + \theta)) \\
&+ 2\sin(2\Theta - 2\theta - \phi) + 2\sin(2\Theta + 2\theta - \phi) + 2\sin(2\Theta - 2\theta + \phi) + 2\sin(2\Theta + 2\theta + \phi)]\} \\
&\times \{\cos^8(\Theta) + \cos^4(\theta)\sin^8(\Theta) - 2\cos^2(\theta)\cos(2\phi)\sin^8(\Theta)\sin^2(\theta) + \sin^8(\Theta)\sin^4(\theta) \\
&+ 2\cos^2(\Theta)\sin^6(\Theta)(-2\cos^4(\theta)\cos(\phi) + 4\cos^3(\theta)\cos^2(\phi)\sin(\theta) + \cos^2(\theta)(3 + \cos(2\phi)) \\
&\times \sin^2(\theta) - 4\cos(\theta)\cos(\phi)\sin^3(\theta) - 2(-1 + \cos(\phi))\sin^4(\theta)) + \frac{3}{32}\sin^4(2\Theta)\sin^2(2\theta) \\
&- 2\cos^7(\Theta)\sin(\Theta)(\cos(2\theta)(1 + \cos(\phi)) + \cos(\phi)\sin(2\theta)) - 2\cos(\Theta)\cos(2\theta)\sin^7(\Theta) \\
&\times (-1 + \cos(\phi)(1 + \sin(2\theta))) + 4\cos^6(\Theta)\sin^2(\Theta)(\cos^4(\theta)(1 + \cos(\phi)) - 4\cos(\theta)\cos^2(\frac{\phi}{2})\cos(\phi)\sin^3(\theta) \\
&+ \cos(\phi)\sin^4(\theta) + \cos^2(\theta)\sin^2(\theta)\sin^2(\phi)) - \frac{1}{8}\cos^4(\Theta)\sin^4(\Theta)(-12 - 4\cos(4\theta) + \cos(4\theta - 2\phi) \\
&- 2\cos(2\phi) + \cos(2(2\theta + \phi))) + 8\sin(4\theta) + 4\sin(4\theta - 2\phi) + 16\sin(2\theta - \phi) + 8\sin(4\theta - \phi) + 16\sin(2\theta + \phi) \\
&+ 4\sin(2(2\theta + \phi)) + 8\sin(4\theta + \phi) + \cos^3(\Theta)\sin^5(\Theta)(-2\cos(2\theta)(-1 + \cos(\phi) + \cos(\phi)(2 + 2\cos(4\theta) \\
&+ \cos(4\theta - \phi) - 2\cos(\phi) + \cos(4\theta + \phi) + 2\sin(2\theta) + \sin(4\theta - \phi) + \sin(4\theta + \phi))) + \cos^5(\Theta)\sin^3(\Theta) \\
&\times (-2\cos(2\theta)(1 + \cos(\phi)) + \cos(\phi)(-2 - 2\cos(4\theta) - \cos(4\theta - \phi) + 2\cos(\phi) - \cos(4\theta + \phi) + \sin(4\theta) \\
&+ \sin(4\theta - \phi) + \sin(4\theta + \phi)))\}, \\
F &= \{1/[8 - \cos(2\Theta - \phi) + \cos(6\Theta - \phi) + \cos(4\Theta - 2\theta - \phi) - \cos(4\Theta + 2\theta - \phi) - \cos(2\Theta + \phi) \\
&+ \cos(6\Theta + \phi) + \cos(4\Theta - 2\theta + \phi) - \cos(4\Theta + 2\theta + \phi) - 2\sin(4\Theta - 2\theta) - 2\sin(2(2\Theta + \theta)) \\
&+ 2\sin(2\Theta - 2\theta - \phi) + 2\sin(2\Theta + 2\theta - \phi) + 2\sin(2\Theta - 2\theta + \phi) + 2\sin(2\Theta + 2\theta + \phi)]\} \\
&\times \{\cos^8(\Theta) + \cos^4(\theta)\sin^8(\Theta) - 2\cos^2(\theta)\cos(2\phi)\sin^8(\Theta)\sin^2(\theta) + \sin^8(\Theta)\sin^4(\theta) \\
&+ 2\cos^4(\Theta)\sin^4(\Theta)(\cos^4(\theta) - 2\cos^3(\theta)(1 - 4\cos(\phi) + \cos(2\phi))\sin(\theta) + \cos^2(\theta)\cos(2\phi)\sin^2(\theta) \\
&+ 4\cos(\theta)\cos^2(\phi)\sin^3(\theta) + \sin^4(\theta)) + 2\cos^2(\Theta)\sin^6(\Theta)(2\cos^4(\theta)\cos(\phi) + 4\cos^3(\theta)\cos^2(\phi)\sin(\theta) \\
&+ \cos^2(\theta)(3 + \cos(2\phi))\sin^2(\theta) + 4\cos(\theta)\cos(\phi)\sin^3(\theta) + 4\cos^2(\frac{\phi}{2})\sin^4(\theta)) + \frac{3}{32}\sin^4(2\Theta)\sin^2(2\theta) \\
&+ 2\cos^7(\Theta)\sin(\Theta)(\cos(2\theta)(-1 + \cos(\phi)) + \cos(\phi)\sin(2\theta)) + 2\cos(\Theta)\cos(2\theta)\sin^7(\Theta)(1 + \cos(\phi) \\
&\times (1 + \sin(2\theta))) - 4\cos^6(\Theta)\sin^2(\Theta)(\cos^4(\theta)(-1 + \cos(\phi)) + \cos(\phi)\sin^4(\theta) - 4\cos(\theta)\cos(\phi)\sin^3(\theta)\sin^2(\frac{\phi}{2}) \\
&- \cos^2(\theta)\sin^2(\theta)\sin^2(\phi)) + \cos^3(\Theta)\sin^5(\Theta)(2\cos(2\theta)(1 + \cos(\phi)) + \cos(\phi)(-2 - 2\cos(4\theta) + \cos(4\theta - \phi) \\
&- 2\cos(\phi) + \cos(4\theta + \phi) - 2\sin(2\theta) + \sin(4\theta - \phi) + \sin(4\theta + \phi))) + \cos^5(\Theta)\sin^3(\Theta)(2\cos(2\theta)(-1 + \cos(\phi)) \\
&+ \cos(\phi)(2 + 2\cos(4\theta) - \cos(4\theta - \phi) + 2\cos(\phi) - \cos(4\theta + \phi) - \sin(4\theta) + \sin(4\theta - \phi) + \sin(4\theta + \phi)))\}, \\
G &= \{1/[8 - \cos(2\Theta - \phi) + \cos(6\Theta - \phi) - \cos(4\Theta - 2\theta - \phi) + \cos(4\Theta + 2\theta - \phi) - \cos(2\Theta + \phi) \\
&+ \cos(6\Theta + \phi) - \cos(4\Theta - 2\theta + \phi) + \cos(4\Theta + 2\theta + \phi) + 2\sin(4\Theta - 2\theta) + 2\sin(4\Theta + 2\theta)) \\
&- 2\sin(2\Theta - 2\theta - \phi) - 2\sin(2\Theta + 2\theta - \phi) - 2\sin(2\Theta - 2\theta + \phi) - 2\sin(2\Theta + 2\theta + \phi)]\} \\
&\times \{\cos^8(\Theta) + \cos^4(\theta)\sin^8(\Theta) - 2\cos^2(\theta)\cos(2\phi)\sin^8(\Theta)\sin^2(\theta) + \sin^8(\Theta)\sin^4(\theta) \\
&+ 2\cos^4(\Theta)\sin^4(\Theta)(\cos^4(\theta) - 4\cos^3(\theta)\cos^2(\phi)\sin(\theta) + \cos^2(\theta)\cos(2\phi)\sin^2(\theta) \\
&+ 2\cos(\theta)(1 - 4\cos(\phi) + \cos(2\phi))\sin^3(\theta) + \sin^4(\theta)) + 2\cos^2(\Theta)\sin^6(\Theta)(2\cos^4(\theta)(1 + \cos(\phi)) \\
&- 4\cos^3(\theta)\cos(\phi)\sin(\theta) + \cos^2(\theta)(3 + \cos(2\phi))\sin^2(\theta) - 4\cos(\theta)\cos^2(\phi)\sin^3(\theta) + 2\cos(\phi)\sin^4(\theta)) \\
&+ \cos(\Theta)\cos(2\theta)\sin^7(\Theta)(-2 + 2\cos(\phi)(-1 + \sin(2\theta))) + \frac{3}{32}\sin^4(2\Theta)\sin^2(2\theta) - 2\cos^7(\Theta)\sin(\Theta) \\
&\times (\cos^2(\theta)(-1 + \cos(\phi)) + \cos(\phi)\sin(2\theta)) - 4\cos^6(\Theta)\sin^2(\Theta)(\cos^4(\theta)\cos(\phi) + (-1 + \cos(\phi))\sin^4(\theta) \\
&+ 4\cos^3(\theta)\cos(\phi)\sin(\theta)\sin^2(\frac{\phi}{2}) - \cos^2(\theta)\sin^2(\theta)\sin^2(\phi)) + \cos^3(\Theta)\sin^5(\Theta)(-2\cos(2\theta)(1 + \cos(\phi)) \\
&+ \cos(\phi)(-2 - 2\cos(4\theta) + \cos(4\theta - \phi) - 2\cos(\phi) + \cos(4\theta + \phi) + 2\sin(2\theta) + \sin(4\theta - \phi) + \sin(4\theta + \phi))) \\
&+ \cos^5(\Theta)\sin^3(\Theta)(-2\cos(2\theta)(-1 + \cos(\phi)) + \cos(\phi)(2 + 2\cos(4\theta) - \cos(4\theta - \phi) + 2\cos(\phi) \\
&- \cos(4\theta + \phi) - \sin(4\theta) + \sin(4\theta - \phi) + \sin(4\theta + \phi)))\},
\end{aligned}$$

and

$$\begin{aligned}
H &= \{1/[8 + \cos(2\Theta - \phi) - \cos(6\Theta - \phi) + \cos(4\Theta - 2\theta - \phi) - \cos(4\Theta + 2\theta - \phi) + \cos(2\Theta + \phi) \\
&- \cos(6\Theta + \phi) + \cos(4\Theta - 2\theta + \phi) - \cos(4\Theta + 2\theta + \phi) + 2\sin(4\Theta - 2\theta) + 2\sin(4\Theta + 2\theta) \\
&+ 2\sin(2\Theta - 2\theta - \phi) + 2\sin(2\Theta + 2\theta - \phi) + 2\sin(2\Theta - 2\theta + \phi) + 2\sin(2\Theta + 2\theta + \phi)]\} \\
&\times \{\cos^8(\Theta) + \cos^4(\theta)\sin^8(\Theta) - 2\cos^2(\theta)\cos(2\phi)\sin^8(\Theta)\sin^2(\theta) + \sin^8(\Theta)\sin^4(\theta) \\
&+ 2\cos^4(\Theta)\sin^4(\Theta)(\cos^4(\theta) - 4\cos^3(\theta)\cos^2(\phi)\sin(\theta) + \cos^2(\theta)\cos(2\phi)\sin^2(\theta) \\
&+ 2\cos(\theta)(1 + 4\cos(\phi) + \cos(2\phi))\sin^3(\theta) + \sin^4(\theta)) + 2\cos^2(\Theta)\sin^6(\Theta)(-2\cos^4(\theta)(-1 + \cos(\phi)) \\
&+ 4\cos^3(\theta)\cos(\phi)\sin(\theta) + \cos^2(\theta)(3 + \cos(2\phi))\sin^2(\theta) - 4\cos(\theta)\cos^2(\phi)\sin^3(\theta) - 2\cos(\phi)\sin^4(\theta)) \\
&- \cos(\Theta)\cos(2\theta)\sin^7(\Theta)(2 + 2\cos(\phi)(-1 + \sin(2\theta))) + \frac{3}{32}\sin^4(2\Theta)\sin^2(2\theta) + 2\cos^7(\Theta)\sin(\Theta) \\
&\times (\cos^2(\theta)(1 + \cos(\phi)) + \cos(\phi)\sin(2\theta)) - 4\cos^6(\Theta)\sin^2(\Theta)(\cos^4(\theta)\cos(\phi) + 4\cos^3(\theta))\cos^2(\frac{\phi}{2}) \\
&\times \cos(\phi)\sin(\theta) + 2\cos^2(\frac{\phi}{2})\sin^4(\theta) + \cos^2(\theta)\sin^2(\theta)\sin^2(\phi)) + \cos^3(\Theta)\sin^5(\Theta)(2\cos(2\theta)(-1 + \cos(\phi)) \\
&+ \cos(\phi)(2 + 2\cos(4\theta) + \cos(4\theta - \phi) - 2\cos(\phi) + \cos(4\theta + \phi) - 2\sin(2\theta) + \sin(4\theta - \phi) + \sin(4\theta + \phi))) \\
&+ \cos^5(\Theta)\sin^3(\Theta)(2\cos(2\theta)(1 + \cos(\phi)) + \cos(\phi)(-2 - 2\cos(4\theta) - \cos(4\theta - \phi) + 2\cos(\phi) - \cos(4\theta + \phi) \\
&+ \sin(4\theta) + \sin(4\theta - \phi) + \sin(4\theta + \phi)))\}.
\end{aligned}$$

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